ON INTERVAL REPRESENTATIONS AND SYMMETRIES OF GRAPHS

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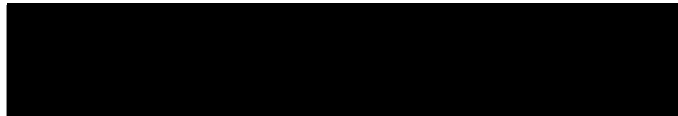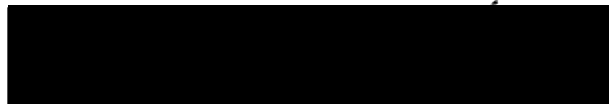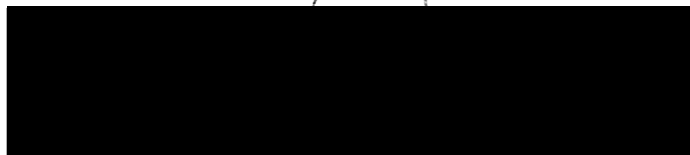
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On Interval Representations and Symmetries of Graphs

Thesis directed by Professor J. Richard Lundgren

ABSTRACT

This thesis focuses on two main topics: interval representations and symmetries of graphs. A graph is *interval* if to every vertex $v \in V(G)$, we can assign an interval of the real line, I_v, such that $xy \in E(G)$ if and only if $I_x \cap I_y \neq \emptyset$. We call the set of intervals of the real line the *interval representation* of the graph. Interval graphs were characterized by the absence of induced cycles larger than 3 and asteroidal triples by Lekkerkerker and Boland [36] in 1962. Subsequently variations on the interval theme have been introduced, including probe interval graphs and the interval p-graph, which is a generalization of the interval bigraph.

A natural extension of interval graphs, called interval bigraphs, were introduced by Harary, Kabell, and McMorris [28] in 1982. A bipartite graph $G = (X, Y)$ is an interval bigraph if to every vertex, $v \in V(G)$, we can assign an interval of the real line, I_v, such that $xy \in E(G)$ if and only if $I_x \cap I_y \neq \emptyset$ and $x \in X$ and $y \in Y$. Initially it was thought that the natural extension of asteroidal triples of vertices to asteroidal triples of edges along with induced cycles larger than 4 would be a forbidden subgraph characterization [28]. However that was not to be, and there is no forbidden subgraph characterization of

interval bigraphs to date. The only forbidden subgraph characterization in the literature is for cycle-free interval bigraphs in [13].

In 2002 Brown, Flink, and Lundgren introduced a further extension of interval bigraphs called interval k-graphs [11]. We change the name to interval p-graphs here to avoid confusion. Let $G = \{X_1, X_2, ..., X_p\}$ be a multipartite graph. A graph G is an *interval p-graph* if there exists an assignment to each vertex, $v \in V(G)$, an interval of the real line, I_v, such that $xy \in E(G)$ if and only if $I_x \cap I_y \neq \emptyset$ and $x \in X_i$, $y \in X_j$ and $i \neq j$.

The forbidden subgraph characterization for interval 2-graphs (interval bigraphs) has proven difficult, hence a forbidden subgraph characterization for interval p-graphs, for $p \geq 2$, will also be difficult, whence we restrict attention to a natural generalization of the cycle-free case: the case of k-trees. The class of k-*trees* is the set of all graphs that can be obtained by the following construction: (i) the k-complete graph, K_k, is a k-tree; (ii) to a k-tree Q' with $n - 1$ vertices $(n > k)$ add a new vertex adjacent to a k-complete subgraph of Q'. In chapter 2, two characterizations of k-tree interval p-graphs are given, one of which is a forbidden subgraph characterization.

Moving to a different variation on the interval property, we next investigate probe interval 2-trees. A graph G is a *probe interval graph* (PIG) if there is a partition of $V(G)$ into sets P and N and a collection $\{I_v \mid v \in V(G)\}$ of intervals of the real line such that, for $u, v \in V(G)$, $uv \in E(G)$ if and only if $I_u \cap I_v \neq \emptyset$ and at least one of u, v belongs to P. Probe interval graphs were invented to help with modeling in the human genome project. Pržulj and Corneil found in [41] that 2-tree probe interval graphs have a large set of forbidden subgraphs,

at least 62. We give a characterization of 2-tree PIGs and add to the list of forbidden subgraphs.

The second concentration of this thesis, symmetries of graphs, began in 1996 with a paper by Albertson and Collins [2]. A coloring of the vertices of a graph G, $f : V(G) \rightarrow \{1, ..., r\}$ is said to be r-*distinguishing* if no nontrivial automorphism of the graph preserves all of the vertex colors. The *distinguishing number* of a graph G is defined by $D(G) = \min \{r \mid G$ has a coloring that is r-distinguishing $\}$. Albertson and Collins determined the distinguishing number for graphs that realize the dihedral group. We let D_n denote the dihedral group of order $2n$, which is the group of symmetries of a regular n-gon. Albertson and Collins [2] proved that if G realizes D_n then $D(G) = 2$ unless $n = \{3, 4, 5, 6, 10\}$ in which case $D(G)$ is either 2 or 3.

Since Albertson and Collins introduced this topic, there has been much investigation into distinguishing colorings. Areas of these investigations include the distinguishing number of Cartesian products ([1], [6], [33], [35]), determining a bound on the distinguishing number ([17], [34]), and the distinguishing number of trees and forests ([15]).

In our investigations we generalize distinguishing colorings to list-distinguishing colorings. Given a family L of lists assigning available colors to the vertices of G, we say that G is L-*distinguishable* if there is a distinguishing coloring f of G such that $f(v) \in L(v)$ for all v. The *list-distinguishing number* of G, written $D_\ell(G)$, is the smallest positive integer k such that G is L-distinguishable for any assignment L of lists with $|L(v)| = k$ for all v. Since all of the lists can be identical, we see that $D(G) \leq D_\ell(G)$.

Here we determine the list-distinguishing number for several families of graphs and highlight a number of distinctions between the problems of distinguishing and list-distinguishing a graph. We first give a Brooks-type result for list-distinguishing colorings. By determining precisely when the distinguishing number is 3 in small cases, we show that the list-distinguishing number is equal to the distinguishing number for all graphs that realize the dihedral group. Lastly, we give the list-distinguishing number for some Cartesian product graphs.

This abstract accurately represents the content of the candidate's thesis. I recommend its publication.

Signed ███████████████████████████████████

J. Richard Lundgren

DEDICATION

I dedicate this thesis to my parents, Doug Flesch and Janna Hertzler. Without their love and support, none of this would have been possible.

ACKNOWLEDGMENT

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FIGURES

Figure

1. Introduction

1.1 Prologue and Definitions

Questions in Graph Theory can often be stated as: "Determine which graphs have property φ." The work in this thesis stems from questions of this structure. We first consider variations on the property of having an interval representation and then move to the property of having list-distinguishing number n.

Notation and definitions will be introduced as needed, but we start with some of the basics. For a comprehensive treatment of Graph Theory, see [49]. Let a *graph* G have vertex set $V(G)$ and edge set $E(G)$. If $x, y \in V(G)$ have an edge between them in G then we say x and y are *adjacent*, which we denote $xy \in E(G)$. If a vertex v is an endpoint of e, we say that e is *incident* to v. If x and y are vertices or edges or a combination thereof, by $x+y$, we mean the graph induced on $x \cup y$. If G is multipartite, we denote the partitions of the vertex set as $V(G) = \{X_1 \cup X_2 \cup \ldots X_p\}$. The set $N(x) = \{v \in V(G) \mid vx \in E(G)\}$ is the *neighborhood* of a vertex x. We will denote $N(xy)$ for $N(x) \cup N(y)$. Given a subset $V' \subset V(G)$, the graph *induced* on V' is denoted $G(V')$ and is the subgraph in which vertices of V' are adjacent in $G(V')$ if and only if they are adjacent in G. A *complete graph* or a *clique* is a graph with every pair of vertices adjacent, and a *complete multipartite graph* is a p-partite graph with every pair of vertices that belong to different partite sets adjacent. A subset of vertices $V' \subset V(G)$ with $G(V')$ having no edges is called an *independent set*. To denote a path from vertex a to c that uses the edges ab and bc we may concatenate

vertices as *abc*. The *degree sequence* of a graph is the list of vertex degrees, where the degree of a vertex is the number of edges incident to that vertex. In this thesis, the degree sequence is listed in non-decreasing order.

A *k-coloring* of a graph G is a labeling $f : V(G) \rightarrow S$, where $|S| = k$. A *k*-coloring is *proper* if adjacent vertices have different labels. A graph is *k-colorable* if it has a proper *k*-coloring. The *chromatic number* $\chi(G)$ is the least k such that G is *k*-colorable. A *k-edge-coloring* of G is a labeling $f : E(G) \rightarrow S$, where $|S| = k$. A *k*-edge-coloring is proper if incident edges have different labels. A graph is *k-edge-colorable* if it has a proper *k*-edge-coloring. The *edge-chromatic number* $\chi'(G)$ of a loopless graph G is the least k such that G is *k*-edge-colorable.

For each vertex v in a graph G let $L(v)$ denote a list of colors available at v. A *list coloring* is a proper coloring f such that $f(v) \in L(v)$ for all v. A graph G is *list k-colorable* if every assignment of *k*-element lists to the vertices permits a proper list coloring. The *list chromatic number* $\chi_\ell(G)$ is the minimum k such that G is list *k*-colorable. Let $L(e)$ denote the list of colors available for an edge e. A *list edge-coloring* is a proper edge-coloring f with $f(e)$ chosen from $L(e)$ for each e. The *edge-choosability* $\chi'_\ell(G)$ is the minimum k such that every assignment of lists of size k yields a proper list edge-coloring.

We say that an order of subgraphs $G_1, \ldots, G_n$ of a graph G is *consecutive* if $u \in V(G_i) \cap V(G_j)$, $i \leq j$, then $u \in V(G_k)$ for all $i \leq k \leq j$. When a consecutive order exists for a family F of subgraphs, we say that F is *consecutively orderable*. A collection of sets is said to have the *Helly property* if whenever a subcollection $S_1, \ldots, S_k$ of them intersect pairwise, then $\cap_{i=1}^k S_i$ is nonempty.

A *breadth-first search (BFS)* starts at a given vertex v, which we call the *root* and is at level 0. In the first stage, we visit all vertices that are at the distance of one edge away, and all of these vertices are placed into level 1. In the second stage, we visit all the new vertices we can reach at the distance of two edges away from v. These new vertices, which are adjacent to the level 1 vertices and are not previously assigned a level, are placed into level 2, and so on. The BFS traversal terminates when every vertex has been visited, and creates a BFS spanning tree of a connected graph G. The *parent* of a vertex x is the vertex immediately following x on the unique path from x to the root v on the BFS tree. Two vertices are siblings if they have the same parent. A *descendant* w of x is any vertex that contains x on the path from w to the root v on the BFS tree. An *ancestor* of x is any vertex on the path from x to the root v on the BFS tree.

An *isomorphism* from a graph G to a graph H is a bijection $f : V(G) \rightarrow V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say G is isomorphic to H, written $G \cong H$, if there is an isomorphism from G to H. An *automorphism* of a graph G is a permutation σ of the vertex set $V(G)$, such that for any edge $e = (u, v)$, $\sigma(e) = (\sigma(u), \sigma(v))$ is also an edge. The set of automorphisms of a given graph, under the operation of composition, forms a group. A *group* is an ordered pair $(G, *)$ where G is a set and $*$ is an operation on G satisfying the following axioms: (1) For all $a, b \in G$, $a * b \in G$, so G is closed under $*$, (2) $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$, so $*$ is associative, (3) there exists an element e in G, called the identity, such that for all $a \in G$ we have $e * a = a * e = a$, and (4) for each $a \in G$ there is an element a^{-1} of G,

called an inverse of a, such that $a * a^{-1} = a^{-1} * a = e$.

A subset S of elements of a group G with the property that every element of G can be written as a product of elements of S and their inverses is called a set of *generators* of G. We denote this as $G = \langle S \rangle$ and say that G is *generated* by S. If H is a nonempty subset of a group G such that H is closed under the binary operation on G and is closed under inverses, we call H a subgroup of G.

Let G be a graph and let Γ be a group. If $Aut(G) \cong \Gamma$ then we will say that G *realizes* Γ. For a vertex v in G, we let $St(v) = \{h \in \Gamma | vh = v\}$ and $O(v) = \{b | v = hb$ for some $h \in \Gamma\}$ be the *stabilizer* and *orbit* of v under the action of Γ on G. The size of the orbit of v under the action of Γ on G is $|Aut(G)|/|St(v)|$, which is referred to as the *Orbit/Stabilizer Theorem*.

A *group action* of a group G on a set A is a map from $G \times A$ to A(written as $g * a$, for all $g \in G$ and $a \in A$) satisfying the following properties: (1) $g_1 * (g_2 * a) = (g_1 * g_2) * a$, for all $g_1, g_2 \in G$, $a \in A$, and (2) $e * a = a$, for all $a \in A$.

1.2 Interval Representations

A graph is *interval* if to every vertex $v \in V(G)$, we can assign an interval of the real line, I_v, such that $xy \in E(G)$ if and only if $I_x \cap I_y \neq \emptyset$. We call the set of intervals of the real line the *interval representation* of the graph. We let I_v denote the interval corresponding the the vertex v and l_v, r_v denote the left and right endpoints of I_v. Interval graphs were introduced in a question posed by Hajós [29] in 1957. In 1959 Benzer used interval graphs to model the fine structure of a gene [5]. This interesting application helped keep interval graphs in the literature and led to variations on the interval property.

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Interval graphs were characterized by the absence of induced cycles larger than 3 and asteroidal triples by Lekkerkerker and Boland [38] in 1962. An *asteroidal triple* (AT) in G is a set A of three vertices such that between any two vertices in A there is a path within G between them that avoids all neighbors of the third. The vertices a, b, and c form an AT in the graph called the 3-sun in Figure 1.2. Other characterizations of interval graphs soon followed in [23] and [25], including a characterization of interval graphs as those that have a consecutive ordering of cliques.

A natural extension of interval graphs, called interval bigraphs, were introduced by Harary, Kabell, and McMorris [30] in 1982. A bipartite graph $G = (X, Y)$ is an *interval bigraph* if to every vertex, $v \in V(G)$, we can assign an interval of the real line, I_v, such that $xy \in E(G)$ if and only if $I_x \cap I_y \neq \emptyset$ and $x \in X$ and $y \in Y$. Interval digraphs, which are related to interval bigraphs, were introduced by Sen, Das, Roy, and West in [46], where they also gave a characterization of interval digraphs (and hence interval bigraphs) based on the adjacency matrix. Interval bigraphs have been studied extensively ([12], [15], [30], [31], [39], and [41]), which includes a consecutive ordering of complete bipartite graphs characterization. Initially it was thought that the natural extension of asteroidal triples of vertices to asteroidal triples of edges along with induced cycles larger than 4 would be another characterization of interval bigraphs [30]. However, Müller [41] found insects and Hell and Huang [31] found edge asteroids and bugs as forbidden subgraphs, and to date the only forbidden subgraph characterization in the literature is for cycle-free interval bigraphs (see Theorem 1.1). Three edges a, c and e of a graph G form an *asteroidal triple of*

edges (ATE) if for any two there is a path from the vertex set of one to the vertex set of the other that avoids the neighborhood of both vertices of the third edge. The edges bx, cz, and ay form an ATE in Figure 1.2. Since it is not pertinent to our studies, we will forgo defining insects, bugs, and edge asteroids in this thesis.

Theorem 1.1 *[15] A cycle-free graph G is an interval bigraph if and only if it has it has no NL10 of Figure 1.1 as an induced subgraph.*
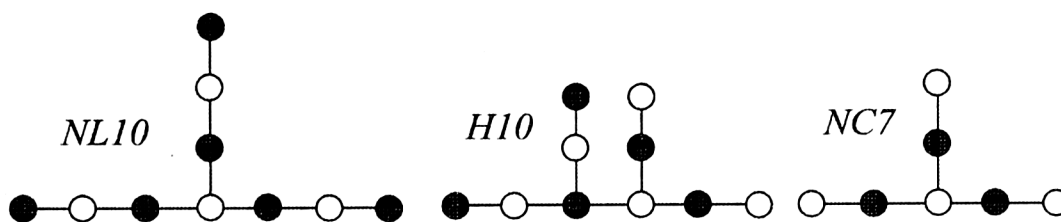
Figure 1.1: The graphs NL10, H10, and NC7.

In 2002 Brown, Flink, and Lundgren introduced a further extension of interval bigraphs called interval k-graphs [13]. We change the name to interval p-graphs here to avoid confusion. Let $G = \{X_1, X_2, ..., X_p\}$ be a multipartite graph. The graph G is an *interval p-graph* if there exists an assignment to each vertex, $v \in V(G)$, an interval of the real line, I_v, such that $xy \in E(G)$ if and only if $I_x \cap I_y \neq \emptyset$ and $x \in X_i$, $y \in X_j$ and $i \neq j$. We call a set of intervals of the real line plus a partition of the vertices an *interval p-representation* of the graph. We consider the vertices of each partite set to have the same color, so adjacency results when two vertices have overlapping intervals and are different colors. The only characterization of interval p-graphs is with a consecutive ordering of complete r-partite subgraphs (see Theorem 1.2).

Theorem 1.2 *[13] A graph is an interval p-graph if and only if there exists a cover of complete r-partite subgraphs that can be consecutively ordered, when $1 \leq r \leq p$ for each subgraph.*
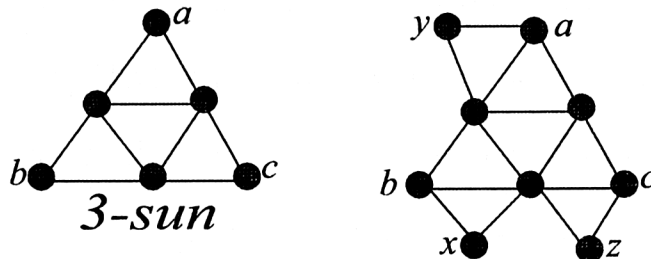
Figure 1.2: On the left is the 3-sun which contains an AT, and on the right is a 2-tree with an ATE.

A forbidden subgraph characterization for interval p-graphs appears to be very difficult as in the case for interval bigraphs, so it seems natural to consider generalizing Theorem 1.1 to the class of graphs called k-trees. The class of k-trees is the set of all graphs that can be obtained by the following construction: (i) the k-complete graph, K_k, is a k-tree; (ii) to a k-tree Q' with $n - 1$ vertices ($n > k$) add a new vertex adjacent to a k-complete subgraph of Q'. Both graphs in Figure 1.2 are 2-trees. Chapter 2 contains a characterization of k-tree interval p-graphs and a forbidden subgraph characterization of 2-tree interval p-graphs.

Through advances in genetics, the need for a new model that was similar to Benzer's original model arose. In 1994 probe interval graphs were introduced by Zhang to model the physical mapping of clones, small subsequences of DNA, for the human genome project ([48], [52], [51], and [50]). A graph G is a *probe interval graph* (PIG) if there is a partition of $V(G)$ into sets P and N and a collection $\{I_v \mid v \in V(G)\}$ of intervals of the real line such that, for $u, v \in V(G)$, $uv \in E(G)$ if and only if $I_u \cap I_v \neq \emptyset$ and at least one of u, v belongs to P.

Intervals of the real line together with a (P, N)-partition will be referred to as a *probe interval representation* of the graph. Considering clones to be vertices and adjacency to be overlap in the sequence of the clones, the model is a probe interval graph with each vertex a probe. However, we do not always have all of the information for the clones or do not want to work with all of the clones at once. Thus a probe interval graph where only the vertices of clones of interest are defined to be probes gives more flexibility in the model.

For our purposes, we consider only the structural properties of probe interval graphs. PIGs and related topics have been studied extensively, including [16], [14], and [27], with many results. McMorris, Wang, and Zhang [40] characterized PIGs as those graphs that have a consecutive ordering of quasi cliques (see [40]). A *quasi clique* Q in a PIG G with a (P, N)-partition is a set of vertices with all vertices of $Q \cap P$ adjacent, and any vertex of $Q \cap N$ adjacent to all vertices of $Q \cap P$.

In 1999 Sheng found in [47] a forbidden subgraph characterization of cycle-free probe interval graphs (see Theorem 1.3), which remains the only forbidden subgraph characterization of PIGs to date. Sheng also found in [47] that the vertex of degree three in an induced NL7 (see Figure 1.1) must be a non-probe in any probe interval representation, which will be used often in this thesis. Moving up from trees, Pržulj and Corneil found in [43] that 2-tree PIGs have a large set of forbidden subgraphs, at least 62. They gave no unifying characteristic of the 62 forbidden subgraphs and made no claim that their list was complete. In chapter 3 we give a characterization of 2-tree PIGs and add to the list of forbidden subgraphs.

Theorem 1.3 *[47] A cycle-free graph is a probe interval graph if and only if it has no induced subgraph isomorphic to $H10$ or $NL10$ in Figure 1.1.*

1.3 Symmetries

Frank Rubin posed the following question in [44] in 1979.

Professor X, who is blind, keeps keys on a circular key ring. Suppose there are a variety of handle shapes available that can be distinguished by touch. Assume that all keys are symmetrical so that a rotation of the key ring about an axis in its plane is undetectable from an examination of a single key. How many shapes does Professor X need to use in order to keep n keys on the ring and still be able to select the proper key by feel?

In 1996 Albertson and Collins [2] framed this question in terms of Graph Theory, which gave rise to the following definitions. A coloring of the vertices of a graph G, $f : V(G) \rightarrow \{1, ..., r\}$ is said to be *r-distinguishing* if no nontrivial automorphism of the graph preserves all of the vertex colors. The *distinguishing number* of a graph G is defined by $D(G) = \min \{r \mid G$ has a coloring that is r-distinguishing $\}$. The professor's key ring can be represented as a cycle and the different handle shapes are vertex colors. As it turns out, three handle shapes are required for five or fewer keys and two handle shapes for six or more keys.

Given the origins of the problem, Albertson and Collins determined the distinguishing number for graphs that realize the dihedral group (see Theorem 4.6). We let D_n denote the dihedral group of order $2n$, which is the group of symmetries of a regular n-gon. We use the standard presentation $D_n = \langle \sigma_n, \tau_n \mid \sigma_n^{2n} = \tau_n^2 = e, \sigma_n \tau_n = \tau_n \sigma_n^{-1} \rangle$ where σ_n and τ_n denote the appropriate

rotation and reflection of the n-gon, respectively. We will write $\tau_n = \tau$ and $\sigma_n = \sigma$ if the context is clear.

Since Albertson and Collins introduced this topic, there has been much investigation into distinguishing colorings. Areas of these investigations include the distinguishing number of Cartesian products ([1], [6], [35], [37]), determining a bound on the distinguishing number ([19], [36]), and the distinguishing number of trees and forests ([17]). The *Cartesian product* of G and H, written $G\Box H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting (u, v) adjacent to (u', v') if and only if (1) $u = u'$ and $vv' \in E(H)$, or (2) $v = v'$ and $uu' \in E(G)$ (see Figure 1.3).
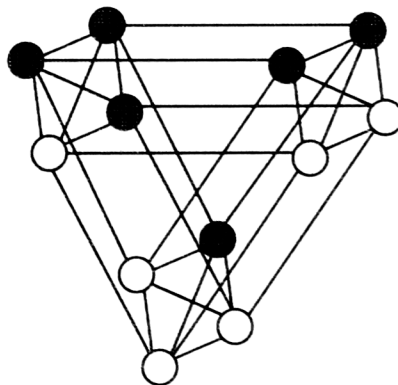
Figure 1.3: A distinguishing coloring of $K_3\Box K_4$.

In our studies we generalize distinguishing colorings to list-distinguishing colorings. Given a family L of lists assigning available colors to the vertices of G, we say that G is *L-distinguishable* if there is a distinguishing coloring f of G such that $f(v) \in L(v)$ for all v. The *list-distinguishing number* of G, written $D_\ell(G)$, is the smallest positive integer k such that G is *L*-distinguishable for any assignment L of lists with $|L(v)| = k$ for all v. Since all of the lists can be

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identical, we see that $D(G) \leq D_\ell(G)$.

Moving from a coloring property to the list-coloring variant has yielded diverse results. Erdős, Rubin, and Taylor found that there are bipartite graphs with arbitrarily large list chromatic number (see Theorem 1.4). Since the chromatic number of a bipartite graph is 2, the chromatic number and the list-chromatic number can be arbitrarily far apart.

Theorem 1.4 ([21]) *If $m = \binom{2k-1}{k}$, then $K_{m,m}$ is not list k-colorable.*

On the other hand, the long-standing List Coloring Conjecture states that $\chi'_\ell(G) = \chi'(G)$ for all G. This conjecture was suggested independently by many researchers, including Vizing, Gupta, Albertson, Collins and Tucker, but it is most often credited to Bollobás and Harris in [7].

These differences in the list variant of a coloring property versus the original coloring property bring us to the central question of our investigations into list-distinguishing colorings.

Question 1.5 *Does there exist a graph G such that $D(G) < D_\ell(G)$?*

In some cases, the list-distinguishing number can readily be shown to be equal to the distinguishing number. For example $D(K_n) = n = D_\ell(K_n)$, since $Aut(K_n) = S_n$. In other cases when $D(G) = D_\ell(G)$, the proof techniques can be significantly different; we consider the proof that $D(C_n) = 2 = D_\ell(C_n)$ when $n \geq 6$ as an illustrative example. The proof that $D(C_n) = 2$ when $n \geq 6$ relies on a clever coloring to distinguish (see Figure 1.4); whereas, the proof that $D_\ell(C_n) = 2$ when $n \geq 6$ requires a more in-depth strategy.
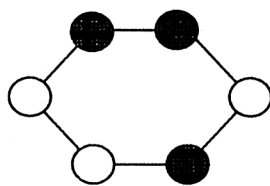
Figure 1.4: A distinguishing coloring of C_6.

Proposition 1.6 *For $n \geq 6$, $D_\ell(C_n) = D(C_n) = 2$.*

Proof: Assign list $L(v)$ such that $|L(v)| = 2$ to each $v \in V(C_n)$. If $|\bigcup L(v)| = 2$, then the lists are identical, so we can color the vertices as the distinguishing coloring, and we are done. Assume then that $|\bigcup L(v)| \neq 2$ and let c be the color that appears the fewest lists. Choose a vertex with c in its list and label it v_1, then continue labeling the vertices consecutively clockwise in numerical order. Since c appears the fewest lists, there are at least $\frac{n}{3}$ vertices that do not have a c in their list. Choose a vertex, v_i, such that $c \notin L(v_i)$ and is not antipodal to v_1. Consider the vertex v_{n+2-i}, which is across from v_i. There exists $x \in L(v_{n+2-i})$ such that $x \neq c$; color v_{n+2-i} with x. Now there is a $y \in L(v_i)$ such that $y \neq x$ and we know that $y \neq c$, so color v_i with y. Now color the remaining vertices with any element from their list that is not c (see Figure 1.5).

Any color-preserving automorphism must map v_1 to v_1, since it is the only vertex colored c. Therefore, either $v_i \mapsto v_{n+2-i}$ or $v_i \mapsto v_i$. In the former case, the automorphism does not preserve the colors. This leaves us with only the latter, which is the trivial automorphism. Therefore the coloring is list-distinguishing, and $D_\ell(C_n) = D(C_n) = 2$.

$\blacksquare$
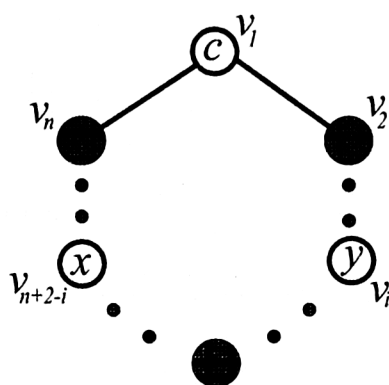
Figure 1.5: The coloring from the proof that $D_\ell(C_n) = 2$ when $n \geq 6$. The black vertices are colored anything from their list that is not c.

In Chapter 4 we give a list-distinguishing Brooks-type theorem and show that $D(G) = D_\ell(G)$ for all graphs that realize the dihedral group. In Chapter 5 we show that $D(G) = D_\ell(G)$ for certain Cartesian product graphs.

2. k-tree Interval p-Graphs

In this chapter we give two characterizations. First we characterize k-trees interval p-graphs as spiny interior k-lobsters. We next use this result to give a forbidden subgraph characterization of 2-tree interval p-graphs.

2.1 Definitions and Background

To describe the structure of k-trees, we use the generalized idea of a path introduced by Beineke and Pippert in [4].

Definition 2.1 *A k-path of G is an alternating sequence of distinct k- and $(k+1)$-cliques of G, $(e_o, t_1, e_1, t_2, e_2, ..., t_p, e_p)$, starting and ending with a k-clique and such that t_i contains exactly two of the distinct k-cliques: e_{i-1} and e_i $(1 \leq i \leq p)$. The length of the k-path is the number p of $(k+1)$-cliques.*

The letters e and t in the definition of a k-path are used to stand for edges and triangles, in the case of 2-trees. This gives one good intuition for the structure of a 2-path, a sequence of distinct edges and triangles. Although there is a unique path of length p up to isomorphism, this is not true for k-paths. For example, Pržulj and Corneil [43] proved that there are three non-isomorphic 2-paths of length five and six non-isomorphic 2-paths of length six (See Figure 2.1).

Drawing further parallels to trees, Proskurowski introduced the notion of a k-caterpillar in [42], and we introduce a spiny interior k-lobster. Before we do this, we classify some of the structure of a k-tree.
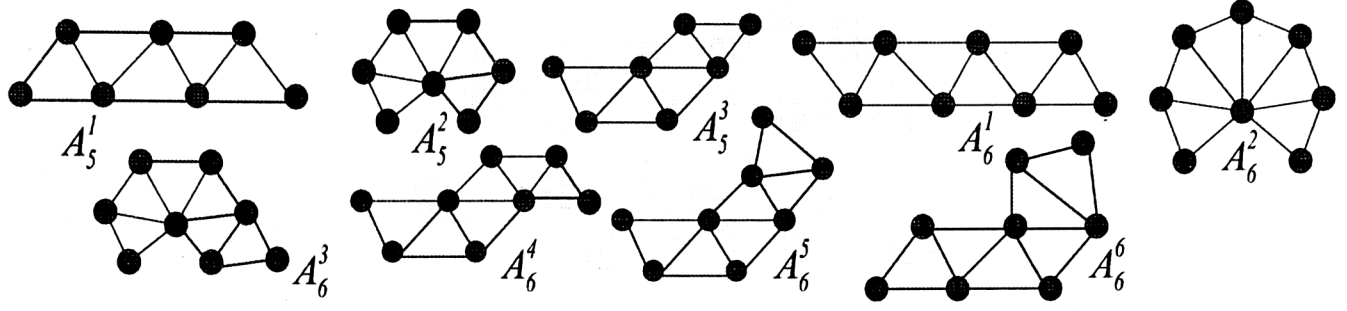
Figure 2.1: The three non-isomorphic 2-paths of length five and six non-isomorphic 2-paths of length six.

A vertex $v \in V(G)$ is a *k-leaf* of G if $N_G(v)$ is a clique. Let G be a k-tree and define G^{1-} to be $G - P_G$, where P_G is the set $\{v \in V(G) : v \text{ is a } k\text{-leaf of G}\}$; iteratively, $G^{2-} = G^{1-} - P_{G^{1-}}$. Suppose G is a k-tree such that G^{2-} is the k-path $(e_0, t_1, e_1, t_2, \ldots, t_p, e_p)$, such that e_0 and e_p are defined in the following way. Let a_0 be a k-leaf of G^{1-} such that $N_{G^{1-}}(a_0) \subset t_1$ and a_p be a k-leaf of G^{1-} such that $N_{G^{1-}}(a_p) \subset t_p$. Define $e_0 = N_{G^{1-}}(a_0)$ and $e_p = N_{G^{1-}}(a_p)$. We will define e_0 and e_p in this way for the rest of the thesis, as it is important for subsequent definitions. We now describe two sets of k-leaves in G and G^{1-} respectively.

- $\partial_1 G = \{v \in V(G) : v \text{ is a } k\text{-leaf of } G, N_G(v) \neq e_i \ (0 \leq i \leq p)\}$;

- $\partial_2 G = \{v \in V(G^{1-}) : v \text{ is a } k\text{-leaf of } G^{1-}, N_{G^{1-}}(v) \neq e_i \ (0 \leq i \leq p)\}$.

In Figure 2.2, G^{2-} is the graph induced by the vertices $\{v_1 \ldots v_6\}$. Also, $e_0 = v_2 v_1$ and $e_4 = v_5 v_6$ because of the 2-leaves of G^{1-} x and z, respectively. For this 2-tree, $\partial_1 G = \{a, b, c, d, f, g, h, j\}$ and $\partial_2 G = \emptyset$, since the 2-leaves of G^{1-} are $\{x, y, z, u\}$, all of which are adjacent to some e_i in G^{2-}.

Definition 2.2 *A 2-caterpillar is a 2-tree G such that G^{1-} is a 2-path. An interior 2-caterpillar is a 2-caterpillar G such that there does not exist a vertex a*
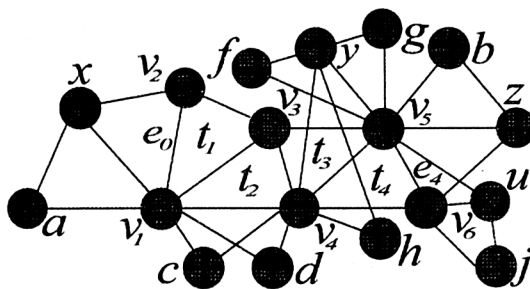
Figure 2.2: A 2-tree with the 3-cliques of G^{2-} labeled.

such that $a \in \partial_1 G$ and $N_G(a) \subset G^{2-}$ and there does not exist vertices $x, y \in \partial_1 G$

such that $N_G(x) \neq N_G(y)$ and $N_{G^{2-}}(x), N_{G^{2-}}(y) \subset e_i, i \in \{0, p\}$.
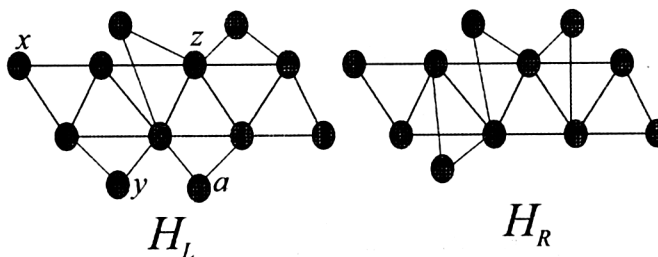
Figure 2.3: H_L is a 2-caterpillar, but not an interior 2-caterpillar, and H_R is an interior 2-caterpillar.

Although H_L and H_R in Figure 2.3 are 2-caterpillars, H_L is not an interior 2-caterpillar for two reasons. First the vertex a is in the set $\partial_1 H_L$ since it is a 2-leaf of H_L and its neighborhood is not an e_i of H_L^{2-}. Furthermore, $N_{H_L}(a)$ is completely contained in H_L^{2-}, thus it violates the first condition of being an interior 2-caterpillar. Moreover the vertices x and y violate the second condition of being an interior 2-caterpillar. They are both in the set $\partial_1 H_L$, their neighborhoods are not the same in H_L, and both neighborhoods in H_L^{2-} are subsets of e_0.

The definition of an interior k-caterpillar is technical, but one can draw on intuition from the word 'interior' for understanding. The graph H_R in Figure 2.3

is an interior 2-caterpillar since all of the 2-leaves are adjacent to the 'interior' edges of the 2-path H_R^{1-}. The first condition of being an interior k-caterpillar says that all k-leaves are adjacent to interior cliques of the k-path G^{2-}. The second condition of being an interior k-caterpillar makes sure that nothing goes wrong at the ends of the k-path G^{2-}, since it deletes two vertices on either end of the longest k-path in the original graph G. This brings us to our next definition.

Definition 2.3 *A graph G is a k-lobster if G^{2-} is a k-path. A spiny interior k-lobster is a k-lobster G with $\partial_2 G = \emptyset$.*
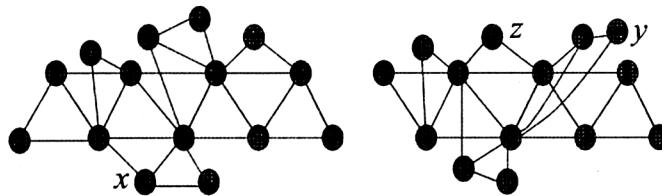
Figure 2.4: On the left is a 2-lobster that is not a spiny interior 2-lobster, and on the right is a spiny interior 2-lobster.

Since caterpillar and lobster are names used to define trees, it seems natural to use the name k-lobster for k-trees. The modifier of interior is put on the k-lobster to signify that all k-paths of length 2 originate from the interior cliques of the k-path G^{2-} in a spiny interior k-lobster (see vertex y in Figure 2.4). The word spiny is added to signify that there could be k-leaves whose adjacency is not an interior clique (see vertex z in Figure 2.4). In the 2-tree on the left in Figure 2.4, $x \in \partial_2 H$, which is why it is not a spiny interior 2-lobster. Another way to define a spiny interior k-lobster is that G is a spiny interior k-lobster if G^{1-} is an interior k-caterpillar.

It is well known that a tree is an interval graph if and only if it is a caterpillar [15]. However, a 2-caterpillar may not be an interval graph since it could contain

an asteroidal triple of vertices (x, y and z in Figure 2.3 are an AT). Eckhoff studied k-trees in the context of extremal interval graphs in [20]. He found that G is a $(k+1)$-extremal interval graph if and only if it is an interior k-caterpillar. Therefore, interior k-caterpillars are the class of k-trees that are interval graphs. We include a simplified proof for completeness.

Theorem 2.4 *If G is an interval k-tree, then G is an interior k-caterpillar.*

Proof: Let G be a k-tree that is not an interior k-caterpillar, and let $P = (e_0, t_1, e_1 \ldots t_p, e_p)$ be a longest k-path in G^{1-} such that e_0 and e_p are the neighborhoods of k-leaves of G. Either G is not a k-caterpillar or it is a k-caterpillar that is not an interior k-caterpillar. If it is not a k-caterpillar then there is some vertex v that is not on P. If it is a k-caterpillar that is not an interior k-caterpillar, then there is some vertex $v \notin P$, adjacent to some k vertices of a t_i such that $e_{i-1}, e_i \neq N_P(v)$. If $i = 1$ or p then the second condition of being a k-caterpillar that is interior is violated; otherwise, the first condition of being a k-caterpillar that is interior is violated. First we consider the case where there is a vertex, $v \notin P$, adjacent to some k vertices of a t_i such that $e_{i-1}, e_i \neq N_P(v)$. If no such v exists, then G is not a k-caterpillar and there are two vertices $a, b \in V(G)$ such that $N_P(a) = e_i$ for $1 \leq i \leq p-1$ and $N_P(b) \subset e_i$ and $a \in N_G(b)$.

Case 1: Assume there exists a vertex, $v \notin P$, adjacent to some k vertices of a t_i such that $e_{i-1}, e_i \neq N_P(v)$. Label z as the vertex not in t_i with $e_{i-1} \subset N_G(z)$, w the vertex not in t_i with $e_i \subset N_G(w)$, and y as the vertex such that $t_i = N_P(v) + y$. We know z and w exist since G is not an interior k-caterpillar and P was the

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longest k-path in G^{1-}. Label $N_P(v) = 1, 2, .., k$ such that z is adjacent to $1, 2, .., k-1$ and w is adjacent to $2, 3, .., k$. Notice that y is one of the vertices labeled as $\{3, \ldots, k-1\}$, and $z, w \in N_G(y)$, since $e_{i-1}, e_i \neq N_P(v)$. The path $z1v$ avoids $N_G(w)$, and zyw avoids $N_G(v)$, and vkw avoids $N_G(z)$ (see G_1 in Figure 2.5). Therefore, $\{w, v, z\}$ is an asteroidal triple, and G is not interval.

Case 2: Assume there are two vertices $a, b \in V(G)$ such that $N_P(a) = e_i$ for $1 \leq i \leq p-1$ and $N_P(b) \subset e_i$ and $a \in N_G(b)$. Let $t_i = z + e_i$, $t_{i+1} = w + e_i$, y be a vertex not in t_i such that $e_{i-1} \subset N_P(y)$, and v be a vertex not in t_i such that $e_{i+1} \subset N_P(v)$. We know that the vertices $\{z, y, w, v\}$ exist since P is the longest k-path in G^{1-} and G is not an interior k-caterpillar. Let $N_P(b) = e_i - c$, and notice that a, z, and w are adjacent to c, since they are each adjacent to all the vertices in e_i. The vertices y and v are both not adjacent to a vertex in e_i, label these vertices as d and e respectively. The path $yzcwv$ avoids $N_G(b)$; the path $yzeab$ avoids $N_G(v)$; and the path $vwdab$ avoids $N_G(y)$ (see G_2 in Figure 2.5). Thus $\{y, v, b\}$ is an asteroidal triple, and G is not interval. ∎
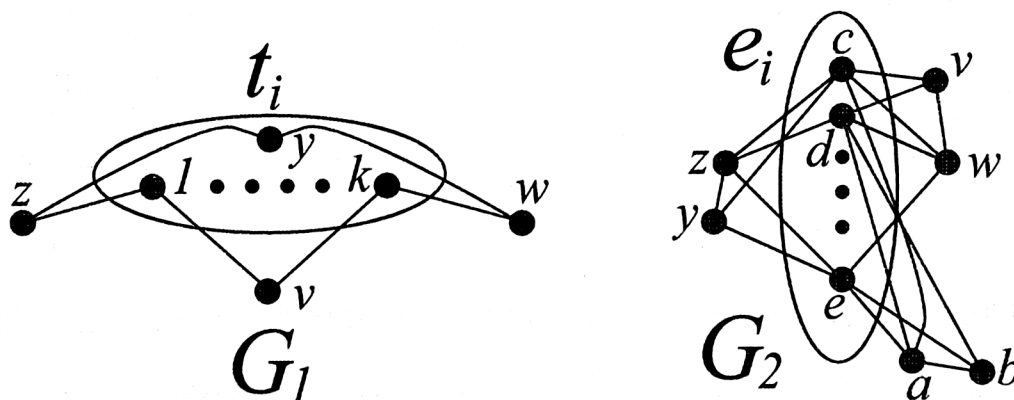
Figure 2.5: Labeled examples for the two cases in Theorem 2.4.

2.2 A Characterization of k-Tree Interval p-Graphs

Now we turn our attention to interval p-graphs. Brown, Flink and Lundgren found in [13] that although interval p-graphs can contain an asteroidal triple, they cannot contain an asteroidal triple of edges. Furthermore, Brown found in [11] that there are graphs that have no interval p-representation and do not contain an ATE (see Figure 2.6).

Theorem 2.5 *[13] If a graph G has an asteroidal triple of edges, then G is not an interval p-graph.*
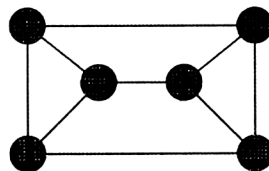
Figure 2.6: An example of a graph that does not have an interval p-representation ($p \geq 3$) and does not contain an ATE.

Using the characterization of cycle-free interval bigraphs and Theorem 2.4 as motivation, one might hope that either k-lobsters or interior k-lobsters characterize k-trees that have an interval p-representation. However, in Figure 2.4 the 2-lobster on the left has an ATE. Interior k-lobsters are interval p-graphs, but do not completely characterize the family. We now show that spiny interior k-lobsters characterize k-trees that have an interval p-representation.

Lemma 2.6 *If G is a k-tree that is not a spiny interior k-lobster, then G contains an asteroidal triple of edges.*

Proof: Let G be a k-tree that it is not a spiny interior k-lobster. Let $P = (e_0, t_1, e_1, ..., t_p, e_p)$ be the longest k-path in G^{2-}, such that e_0 and e_p are the neighborhoods of k-leaves of G^{1-}. It follows from the definition of a spiny

interior k-lobster that either there exists a vertex $a \in V(G^{2-}) - P$ or there exists a vertex $a \in \partial_2 G$. We will first consider the case that there is a vertex $a \in V(G^{1-})$ such that $N_P(a) \subset t_i$, but $e_{i-1}, e_i \neq N_P(a)$. This a could be in $\partial_2 G$ or it could be in $V(G^{2-}) - P$. Thus the only remaining way that G could not be a spiny interior k-lobster is if there exists $a \in V(G^{2-}) - P$ such that $N_P(a) = e_i$ for some $1 \leq i \leq p - 1$.

Case 1: There exists a vertex $a \in V(G^{1-})$ such that $N_P(a) \subset t_i$, but $e_{i-1}, e_i \neq N_P(a)$.

Label some vertices in t_i as follows: $t_i = e_{i-1} + m$, $t_i = e_i + n$, and $t_i = N_P(a) + y$. Notice from the definition of a k-path and our choice of vertex a that m, n, and y are distinct vertices. Label the vertices $z, w \in V(G^{1-})$ such that $z \notin t_i$ and $e_{i-1} \subset N_P(z)$ and such that $w \notin t_i$ and $e_i \subset N_P(w)$; label $r, v \in V(G)$ such that $z, g \in N_P(r)$ where g is any vertex of e_{i-1} and $w, f \in N_P(v)$ where f is any vertex of e_i. We know that the vertices $\{r, z, w, v\}$ exist, since G^{2-} is missing two vertices on either end of the longest k-path in G. Moreover since the vertex a is in $V(G^{1-})$, there must be a vertex $b \notin P$ such that $N_P(b) \subset t_i$ and $a \in N_G(b)$. We claim that the edges $\{rz, ab, wv\}$ form an ATE (see G_1 in Figure 2.7). The path $rzywv$ avoids $N_G(ab)$; the path $rznab$ avoids $N_G(wv)$; and the path $vwmab$ avoids $N_G(rz)$.

Case 2: There exists a vertex $a \in V(G^{2-}) - P$ such that $N_P(a) = e_i$ for some $1 \leq i \leq p - 1$.

Since P is the longest path in G^{2-}, the cliques t_i and t_{i+1} must exist on P, so label $t_i = z + e_i$ and $t_{i+1} = w + e_i$. Let $r \in V(G^{1-})$ such that $z, g \in N_P(r)$ where g is any vertex in e_i, and let $v \in V(G^{1-})$ such that $w, f \in N_P(v)$ where

f is any vertex in e_i. Notice that there must be a vertex in e_i that is not in the neighborhood of r, so label this vertex d, similarly, there must be a vertex in e_i that is not in the neighborhood of v, so label this vertex e. Now let $s \in V(G)$ such that $r, g \in N_G(s)$ where g is any vertex in t_i, and let $y \in V(G)$ such that $v, f \in N_G(y)$ where f is any vertex in t_{i+1}. Again we know that the vertices $\{r, s, y, v\}$ exist, since G^{2-} is missing two vertices on either end of the longest k-path in G.

The vertex a is in $V(G^{2-}) - P$, so there must be a vertex $b \in V(G^{1-})$ such that $N_P(b) \subset e_i$ and $a \in N_G(b)$. Not all of e_i can be in the neighborhood of b, so let $c \in e_i$ such that $c \notin N_G(b)$. The vertices c, d and e need not be distinct. Furthermore, there is also a vertex $h \in V(G)$ such that $N_P(h) \subset e_i$ and $b \in N_G(h)$. We claim that the set $\{rs, vy, bh\}$ is an ATE (see G_2 in Figure 2.7). The path $srzcwvy$ avoids $N_G(bh)$; the path $srzeabh$ avoids $N_G(vy)$; and the path $hbadwvy$ avoids $N_G(rs)$. ∎
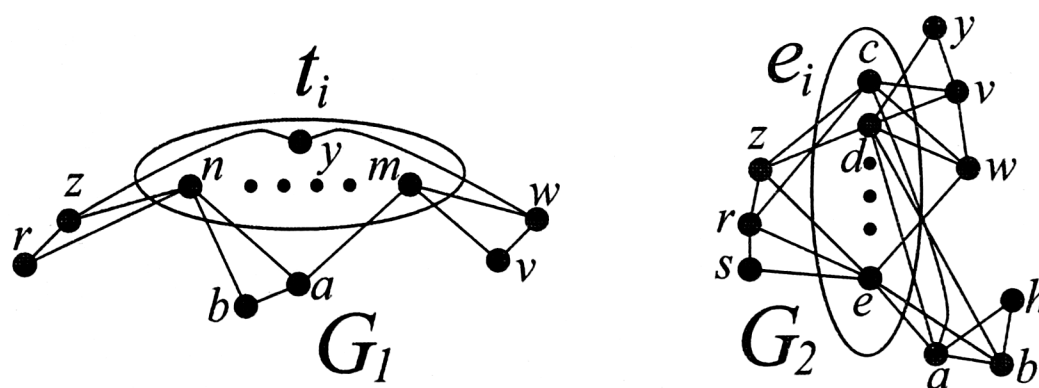
Figure 2.7: Labeled examples for the two cases in Lemma 2.6.

We now give the main result of this section.

Theorem 2.7 *A k-tree G is an interval p-graph if and only if it is a spiny interior k-lobster.*

Proof: Let G be a k-tree that has an interval p-representation, and assume for contradiction that G is not a spiny interior k-lobster. By Lemma 2.6, G contains an asteroidal triple of edges. By Theorem 2.5, interval p-graphs can not contain an ATE, which is a contradiction to our assumption.

Suppose G is a spiny interior k-lobster. Let $G^{2-} = (e_0, t_1, e_1, ..., t_p, e_p)$, where e_0 and e_p are the neighborhoods of some k-leaves of G^{1-}, g and h. Label $t_0 = e_0 + g$ and $t_{p+1} = e_p + h$. Since g and h are k-leaves of G^{1-}, there must be k-leaves of G, a and b, such that $a \in N_G(g)$ and $b \in N_G(h)$. Label $t_{-1} = e_{-1} + a = N_G(a) + a$ and $t_{p+1} = e_{p+1} + b = N_G(b) + b$. Lastly, let e_{-2} be any subset of t_{-1} of size k that contains a, and let e_{p+2} be any subset of t_{p+1} of size k that contains b. The sequence of k- and $k+1$-cliques $(e_{-2}, t_{-1}, e_{-1}, t_0, e_0, t_1, e_1, ..., t_p, e_p, t_{p+1}, e_{p+1}, t_{p+2}, e_{p+2})$ is a longest k-path in G and we call it P.

For each vertex $v \in P$ assign an ordered pair (x, y) such that t_x is the first clique that contains v and t_y is the last. To each vertex assign the interval $I_v = (x, y + \frac{1}{2})$. Assign the colors $1, 2, ..., k+1$ to the vertices of t_{-1}, and assign colors to the rest of the body as follows. If $t_i = e_{i-1} + a_i$ and $t_{i-1} = e_{i-1} + b_{i-1}$, assign a_i the same color as b_{i-1}. For the $(k+1)$-clique t_i, the vertices' intervals intersect at $(i, i + \frac{1}{2})$ and are all different colors, and intervals for the vertices of each e_i intersect at $(i + \frac{1}{2}, i + 1)$ and are all different colors. For each t_i, there are $k - 1$ possible unique neighborhoods for a k-leaf, $w_{i,1}, ..., w_{i,k-1}$, such that $N(w_{i,j}) \subset t_i$, but $N(w_{i,j}) \neq e_i$ or e_{i-1}. For each unique neighborhood, there

may be many k-leaves, which all are assigned the same label. Assign the interval $w_{i,j} = (i + \frac{j-1}{2(k-1)}, i + \frac{j}{2(k-1)})$ to each of these k-leaves. Each k-leaf is adjacent to k of the $k + 1$ vertices in the clique t_i, so let $h_{i,j} = t_i - N(w_{i,j})$. Color $w_{i,j}$ the same as $h_{i,j}$. None of the intervals for the k-leaves $w_{i,1}, \ldots, w_{i,k-1}$ intersect, so no adjacencies result between them. Each k-leaf is the color of the vertex that is not in its neighborhood. So although each k-leaf's interval intersects all the intervals of the clique, adjacencies only result amongst each k-leaf and its k neighborhood in the clique. Furthermore, leaves with a common label are the same color, so there is no resulting adjacency from their overlapping intervals.

There may be many k-paths of length one or two originating from each e_i. Let there be n vertices not on P adjacent to the k vertices of some e_i, and label them $z_{i,1} \ldots z_{i,n}$. To each assign the interval $z_{i,j} = (i + \frac{1}{2} + \frac{j-1}{2n}, i + \frac{1}{2} + \frac{j}{2n})$. Let $t_i = e_i + b_i$ and color $z_{i,j}$, $1 \leq j \leq n$ the same as b_i. For each $z_{i,j}$ there are k possible k-leaves, $m_{i,j,1}, \ldots, m_{i,j,k}$, such that $z_{i,j} \in N(m_{i,j,\ell})$ and $N(m_{i,j,\ell}) \subset e_i + z_{i,j}$. Again, there may be many different k-leaves of this type that have the same label. Assign the interval $m_{i,j,\ell} = (i + \frac{1}{2} + \frac{j-1}{2n} + \frac{\ell-1}{2nk}, i + \frac{1}{2} + \frac{j-1}{2n} + \frac{\ell}{2nk})$. If $g_{i,j,\ell} = e_i + z_{i,j} - N(m_{i,j,\ell})$, color $m_{i,j,\ell}$ the same as $g_{i,j,\ell}$. None of the intervals for the k-leaves intersect, so no adjacencies result between the leaves. Each leaf is adjacent to k of the $k + 1$ vertices of $e_i + z_{i,j}$ and is the color of the vertex that is not in its neighborhood, so the desired adjacencies result. Again, the k-leaves with the same label are the same color, so there is no resulting adjacency from their overlapping intervals. Therefore, we have a p-interval representation for the spiny interior k-lobster. $\blacksquare$
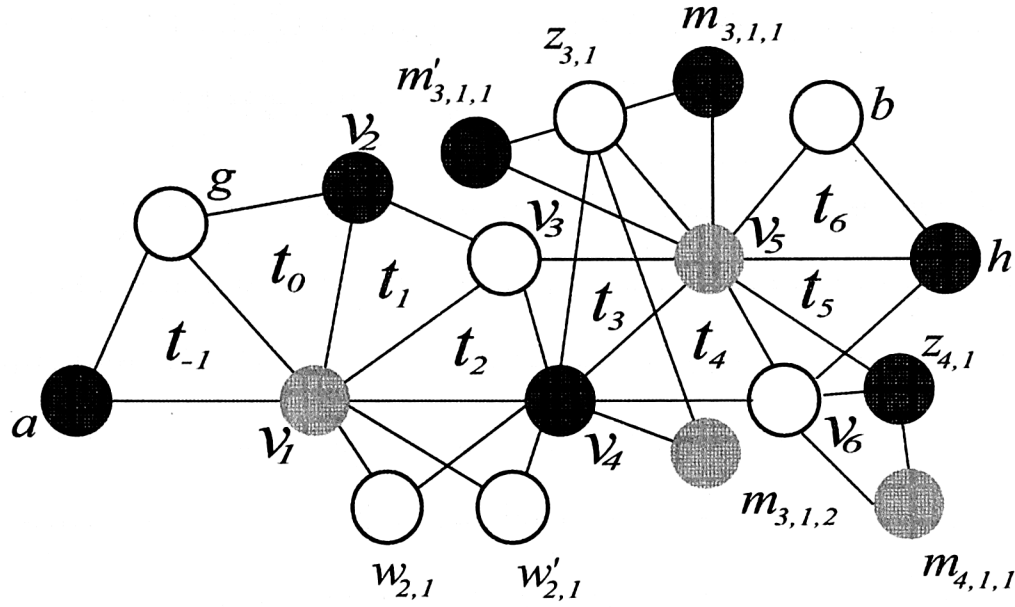
Figure 2.8: An example of a spiny interior 2-lobster that is labeled as in Theorem 2.7.

Figure 2.8 gives a spiny interior 2-lobster with the appropriate labels as in the proof of Theorem 2.7. In the proof, we extend G^{2-} arbitrarily by two cliques on both ends; notice that either vertex, $z_{4,1}$ or h, could have received the $'h'$ label. Figure 2.9 gives the interval p-representation for the graph in Figure 2.8 based on the proof above.
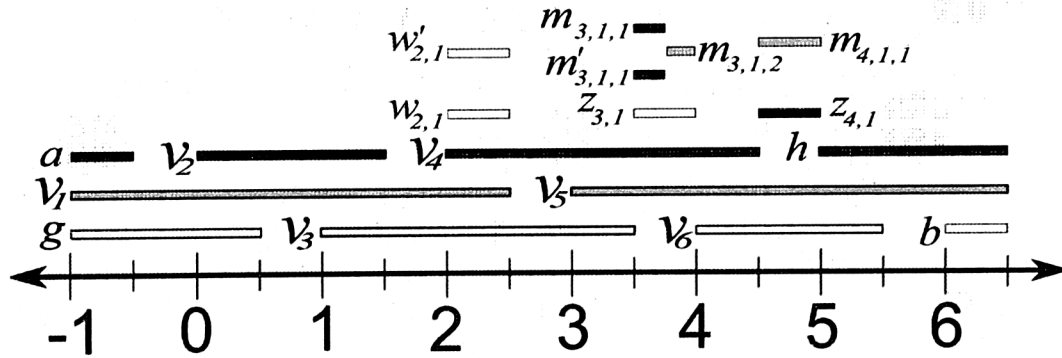
Figure 2.9: The interval p-representation of the 2-tree in Figure 2.8.

Combining the results from Lemma 2.6 and Theorem 2.7, we get the following corollary.

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Corollary 2.8 *A k-tree G is a spiny interior k-lobster if and only if G does not contain an asteroidal triple of edges.*

The relationship between k and p in the previous theorem is a topic of interest. In the proof, we use $k+1$ colors for the interval p-assignment. This is the minimal number of colors that can be used, since K_{k+1} is a subgraph of any non-trivial k-tree. In fact, interval p-graphs have been shown to be perfect in [11], and since collections of intervals have the Helly property, there is never a need for more colors than the size of the largest clique in the graph. However, it is possible to use more colors if there is an advantage. For example, we could have used the color red instead of white on the vertex $z_{3,1}$ in Figure 2.8. In fact when G is a spiny interior k-lobster, we can use a different color for each vertex of G^{1-} since it is a interior k-caterpillar, which has an interval representation. This is why we left the statement of this theorem in terms of k and p. Thus we conclude this section with the following corollary.

Corollary 2.9 *A k-tree G is an interval $(k+1)$-graph if and only if G is a spiny interior k-lobster.*

2.3 Forbidden Subgraph Characterization for 2-Tree Interval p-Graphs

Next we use Theorem 2.7 to find a forbidden subgraph list for 2-tree interval p-graphs. Recall that Pržulj and Corneil found in [43] that 2-trees PIGs have a large set of forbidden subgraphs, at least 62. For k-trees interval p-graphs the forbidden subgraph list also appears to be prohibitively long, so we restrict our
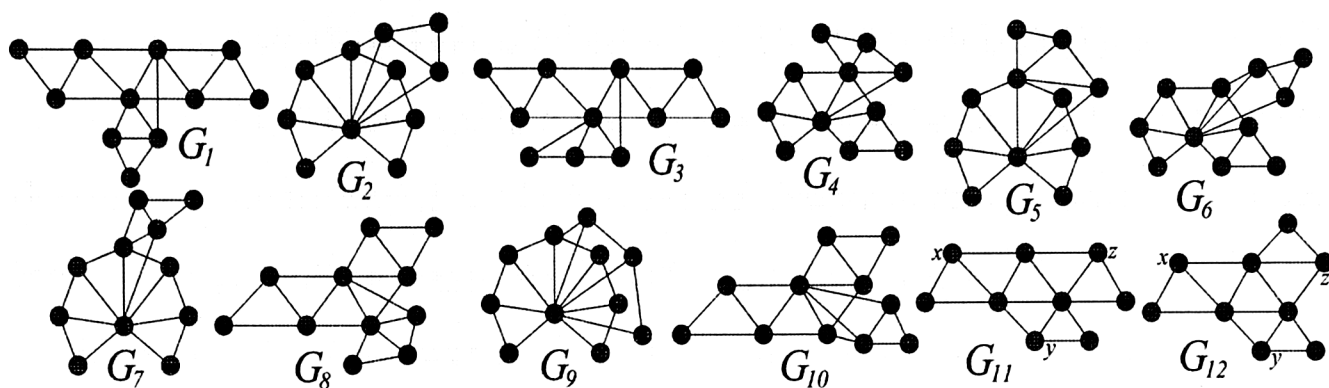
Figure 2.10: Forbidden subgraphs for 2-trees interval p-graphs.

consideration to 2-trees interval p-graphs.

Theorem 2.10 *A 2-tree G has an interval p-representation if and only if G contains no subgraph isomorphic to G_i, $1 \leq i \leq 12$ in Figure 2.10.*

Proof: Let G be a 2-tree that has an interval p-representation. By Theorem 2.7, G is a spiny interior 2-lobster. Consider the 2-trees G_i, $1 \leq i \leq 12$ in Figure 2.10. The graphs G_i^{2-}, $1 \leq i \leq 10$ are not 2-paths, so G_i, $1 \leq i \leq 10$ are not 2-lobsters. In both G_{11} and G_{12}, we have x, y, and z as the 2-leaves of G_{11}^{1-} and G_{12}^{1-}, each of which has a different neighborhood in G^{2-}. We use two of these to define e_0 and e_p and the third vertex is in $\partial_2 G_{11}$ or $\partial_2 G_{12}$, thus G_{11} and G_{12} are not spiny interior 2-lobsters. Therefore, G contains no subgraph isomorphic to G_i, $1 \leq i \leq 12$.

Assume that G does not have an interval p-representation. By Theorem 2.7, G is not a spiny interior 2-lobster. Let $P = (e_0, t_1, e_1, ..., t_p, e_p)$ be the a longest 2-path in G^{2-}, such that e_0 and e_p are the neighborhoods of 2-leaves of G^{1-}. It follows from the definition of a spiny interior 2-lobster that either there exists a vertex $a \in V(G^{2-}) - P$ or there exists a vertex $a \in \partial_2 G$. We will first

consider the case that there is a vertex $a \in V(G^{1-})$ such that $N_P(a) \subset t_i$, but $e_{i-1}, e_i \neq N_P(a)$. This a could be in $\partial_2 G$ or it could be in $V(G^{2-}) - P$. Thus the only remaining way that G could not be a spiny interior 2-lobster is if there exists $a \in V(G^{2-}) - P$ such that $N_P(a) = e_i$ for $1 \leq i \leq p - 1$.

Case 1: There is a vertex $a \in V(G^{1-})$ such that $N_P(a) \subset t_i$, but $e_{i-1}, e_i \neq N_P(a)$.

Label the vertices of t_i such that $t_i = e_{i-1} + x$, $t_i = e_i + y$, and $N_P(a) = t_i - z$. Thus $e_{i-1} = z + y$, $e_i = x + z$, and $N_P(a) = x + y$. For e_{i-1} and e_i to be defined as such there must be vertices b, c distinct from x, y or z such that $e_{i-1} \subset N_P(b)$ and $e_i \subset N_P(c)$. Now the graph induced by the vertices $\{a, b, c, x, y, z\}$ is a 3-sun, and is a subgraph of G^{1-}. Thus there must be vertices $m, n, q \in V(G)$ such that either $b, z \in N_G(m)$ or $b, y \in N_G(m)$ and $c, x \in N_G(n)$ or $c, z \in N_G(n)$ and $y, a \in N_G(q)$ or $x, a \in N_G(q)$. Without loss of generality, we start by adding q such that $y, a \in N_G(q)$. Now, there are 4 possibilities for adding m and n as illustrated by H_i, $1 \leq i \leq 4$ in Figure 2.11. One of these graphs H_i, $1 \leq i \leq 4$ must be a subgraph of G in this case. The graphs H_1, H_2, and H_4 are isomorphic to G_{11}, and the graph H_3 is isomorphic to G_{12}.
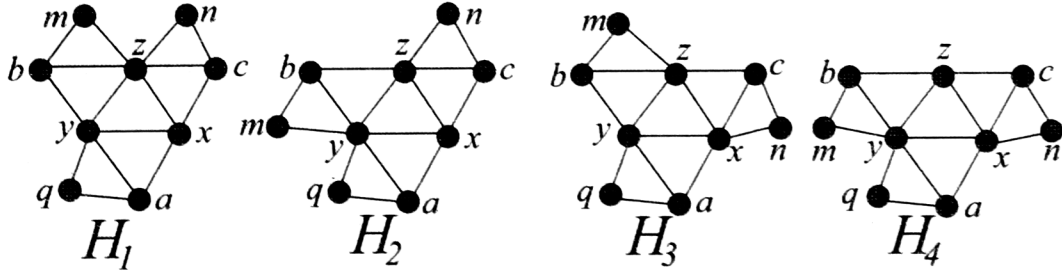
Figure 2.11: The four graphs for Case 1 in Theorem 2.10.

Case 2: There exists a vertex $a \in V(G^{2-}) - P$ such that $N_P(a) = e_i$ for $1 \le i \le p - 1$.

Let $e_i = b + c$. Since P is the longest path in G^{2-}, the cliques t_i and t_{i+1} must exist on P, so label $t_i = z + e_i$ and $t_{i+1} = w + e_i$. Let $r \in V(G^{1-})$ such that $z, g \in N_P(r)$ where g is either b or c, and let $v \in V(G^{1-})$ such that $w, f \in N_P(v)$ where f is either b or c. Now let $s \in V(G)$ such that $r, g \in N_G(s)$ where g is any vertex in t_i, and let $y \in V(G)$ such that $v, f \in N_G(y)$ where f is any vertex in t_{i+1}. The vertices $\{r, s, y, v\}$ exist, since G^{2-} is missing two vertices on either end of any longest k-path in G. The graph induced by the vertices $\{s, r, z, c, b, w, v, y\}$ is a 2-path of length 6 that must be a subgraph of the longest 2-path of G. By Pržulj and Corneil [43] that there are six non-isomorphic 2-paths of length 6 (see Figure 2.1).

The vertex a is in $V(G^{2-}) - P$, so there must be a vertex $b \in V(G^{1-})$ such that $N_P(b) \subset e_i$ and $a \in N_G(b)$. Furthermore, there is also a vertex $h \in V(G)$ such that $N_P(h) \subset e_i$ and $b \in N_G(h)$. Thus there are two possible adjacencies for b, besides a, and two possible adjacencies for h, besides b. Hence, there are four possible configurations of b and h on each of the six non-isomorphic 2-paths of length 6, creating 24 possible configurations, one of which must be a subgraph of G. These 24 possibilities are drawn as H_i, $5 \le i \le 28$ in Figure 2.12. However, $H_6 \cong H_7 \cong H_{26} \cong G_1$, $H_{12} \cong H_{16} \cong G_2$, $H_5 \cong H_8 \cong H_{22} \cong G_3$, $H_{13} \cong H_{18} \cong H_{20} \cong H_{24} \cong G_4$, $H_9 \cong H_{17} \cong H_{19} \cong G_5$, $H_{14} \cong H_{15} \cong H_{28} \cong G_6$, $H_{10} \cong H_{21} \cong G_7$, $H_{23} \cong H_{25} \cong G_8$, $H_{11} \cong G_9$, and $H_{27} \cong G_{10}$. Therefore, if G does not have an interval p-representation, then it contains a subgraph isomorphic to G_i, $1 \le i \le 12$.
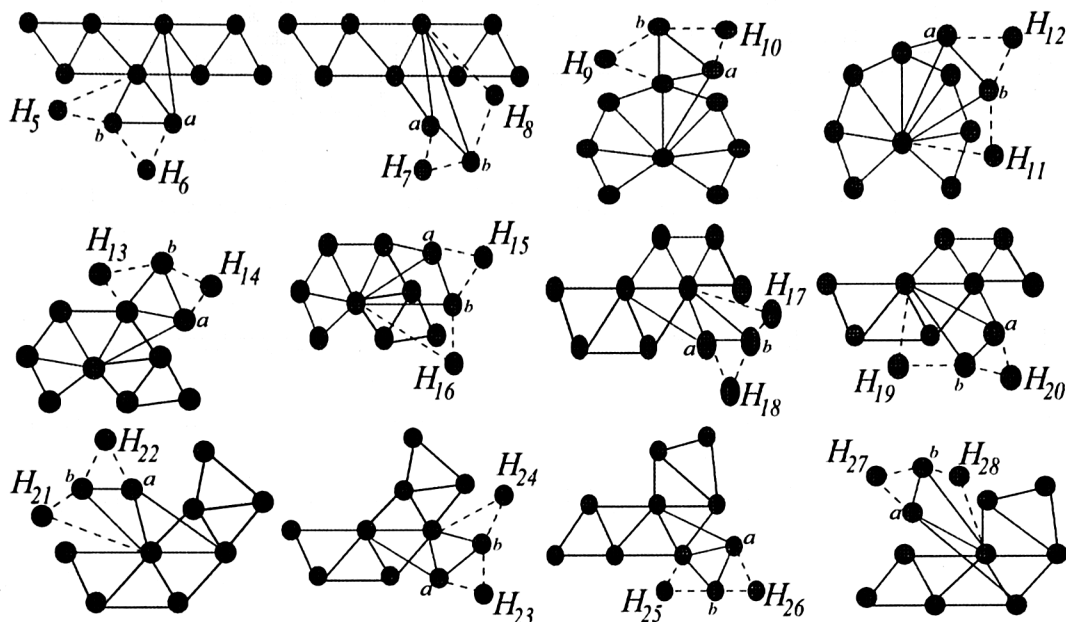
Figure 2.12: Possible forbidden subgraphs for 2-tree interval p-graphs from Case 2 in Theorem 2.10.

The graphs G_{11} and G_{12} are minimal; that is, the removal of any vertex creates an interval p-graph. Figure 2.13 gives two examples of G_{12} with a vertex removed and the corresponding interval p-representation. Notice that $G_{12} - c$ is not a 2-tree, but $G_{12} - e$ is a 2-tree.
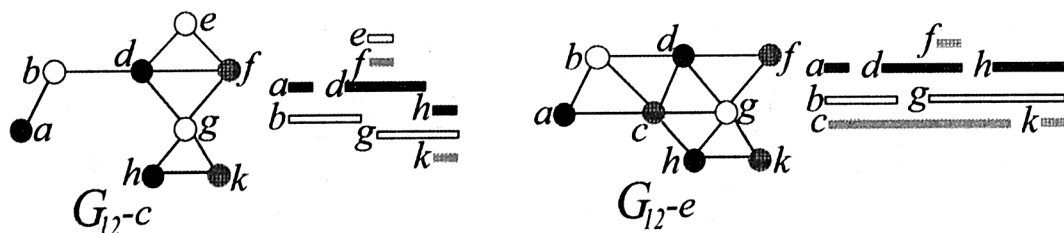
Figure 2.13: Two examples of G_{12} without a vertex and the corresponding interval p-representation.

However, graphs G_i, $1 \leq i \leq 10$ are not minimal in this traditional sense. Each graph contains a vertex v such that the removal of the vertex leaves a graph that still does not have an interval p-representation. Figure 2.14 gives the first five forbidden subgraphs missing one vertex. Each of these still does not have

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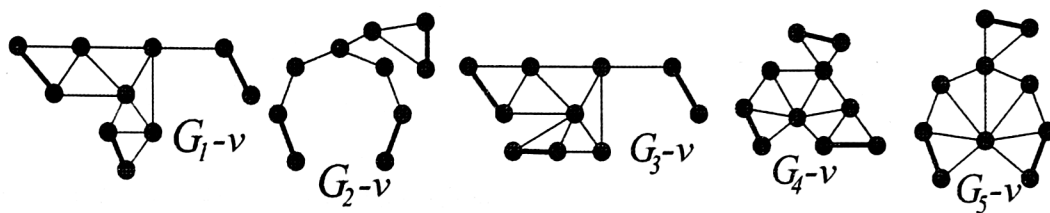
Figure 2.14: Examples of forbidden subgraphs without a vertex that still contain an ATE.

a interval p-representation since the three edges in bold form an ATE in each graph. However, the removal of each of these vertices leaves a subgraph that is not a 2-tree. If we were to remove a 2-leaf from any graph of G_i, $1 \leq i \leq 10$ the resulting graph would be a spiny interior 2-lobster and hence have an interval p-representation. Thus they are minimal in the sense that the removal of any vertex creates a subgraph that is either an interval p-graph or is not a 2-tree. Minimality will be of more importance in Chapter 3.

3. 2-tree Probe Interval Graphs

In this chapter we build on the structure of the spiny interior k-lobster introduced in the last chapter to characterize 2-tree probe interval graphs. The characterization is then used to show the list of 62 forbidden subgraphs obtained by Pržulj and Corneil in [43] is not complete.

3.1 Structure of Probe Interval 2-trees

We let $k = 2$ in all of the definitions introduced in the second chapter. Although spiny interior 2-lobsters characterize the 2-trees that are interval p-graphs, the same is not true for 2-trees that are probe interval graphs. Figure 3.1 gives three forbidden subgraphs for probe interval 2-trees in [43], S_{25}, S_{13} and S_{19}. Each of S_{25}^{2-}, S_{13}^{2-}, and S_{19}^{2-} are 2-paths. In addition the 2-leaves of S_{13}^{1-} and S_{19}^{1-} are $\{x, y, z\}$ and the 2-leaves of S_{25} are $\{y, z\}$, none of which are in $\partial_2 S_{13}$, $\partial_2 S_{19}$, nor $\partial_2 S_{25}$, so all three graphs are spiny interior 2-lobsters. However every PIG is an interval p-graph, so we look for a subset of spiny interior 2-lobsters that characterize 2-tree PIGs. This leads us to more definitions.
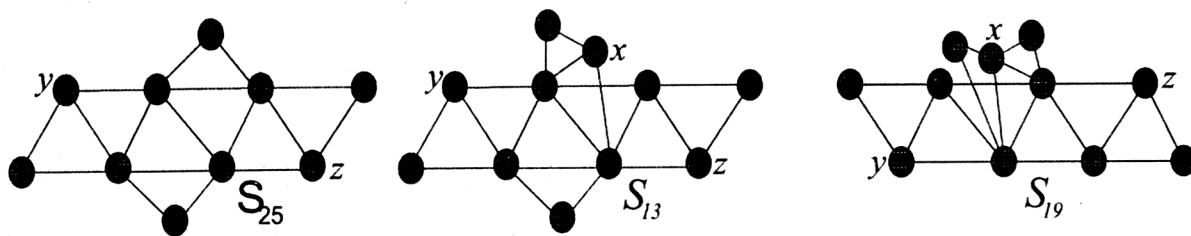
Figure 3.1: Three spiny interior 2-lobsters that are not probe interval graphs in [43].

Suppose G is a 2-tree such that G^{2-} is the 2-path $(e_0, t_1, e_1, t_2, \ldots, t_p, e_p)$. We classify the 2-leaves in $\partial_1 G$ as follows:

- $\partial_1^1 G = \{v \in \partial_1 G : N_G(v) \subseteq V(G^{2-})\}$;

- $\partial_1^2 G = \{v \in \partial_1 G : N_G(v) \not\subseteq V(G^{2-})\}$.

Since the non-adjacencies of these 2-leaves are of particular importance to us, we label some of the vertices of G^{2-} as follows:

- $W^1(G) = \{v \in V(G^{2-}) : N_G(x) = t_i - v \text{ for some } x \in \partial_1^1 G, \ (1 \leq i \leq p)\}$;

- $W^2(G) = \{v \in V(G^{2-}) : N_G(y) = N_{G^{2-}}(z) + z - v = e_i + z - v \text{ for some } y \in \partial_1^2 G \text{ and } z \in V(G), \ (1 \leq i \leq p-1)\}$;

- $W^3(G) = \{v \in V(G^{2-}) : N_G(y) = N_{G^{2-}}(z) + z - v = e_i + z - v \text{ for some } y \in \partial_1^2 G \text{ and } z \in V(G), \ (i \in \{0, p\})\}$;

- $W^{3'}(G) = \{v \in W^3 : N_G(y) = N_{G^{2-}}(z) + z - v = e_i + z - v \text{ and } N_G(s) = N_{G^{2-}}(r) + r - v = e_i + r - v \text{ for } y, s \in \partial_1^2 G \text{ and distinct } z, r \in V(G), \ (i = 0 \text{ or } i = p)\}$;

- $W(G) = W^1(G) \cup W^2(G) \cup W^3(G)$.

In Figure 3.2, $\partial_1^1 G = \{x, v\}$ and $\partial_1^2 G = \{r, s, y, z\}$. Since $N_G(x) = t_1 - w_1$ and $N_G(v) = t_4 - w_3$, $W^1(G) = \{w_1, w_3\}$. Since $N_G(y) = N_{G^{2-}}(b) + b - w_2 = e_2 + b - w_2$ and $N_G(z) = N_{G^{2-}}(c) + c - w_3 = e_3 + c - w_3$, $W^2(G) = \{w_2, w_3\}$. Lastly since $N_G(s) = N_{G^{2-}}(a) + a - w_1 = e_0 + a - w_1$ and $N_G(r) = N_{G^{2-}}(d) + d - w_3 = e_5 + d - w_3$, $W^3(G) = \{w_1, w_3\}$, but $W^{3'}(G) = \emptyset$. Notice that W^3 will never be empty, because G^{2-} eliminates two vertices on either end of the longest 2-path in G. This brings us to our last definition pertaining to the structure of 2-trees.

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Figure 3.2: A spiny interior 2-lobster with the vertices and 3-cliques of G^{2-} labeled.

Definition 3.1 *Let G be a spiny interior 2-lobster with G^{2-} the 2-path $(e_0, t_1, e_1, t_2, \ldots, t_p, e_p)$. If the following four conditions all hold, then G is a sparse spiny interior 2-lobster (ssi2-lobster).*

1. *No $t_i, 1 \leq i \leq p$, has two vertices in $W - W^3$.*

2. *No $t_i, 1 \leq i \leq p$, has two vertices x and y such that $x \in W - W^3$ and $y \in W^{3'}$.*

3. *No $t_i, 1 \leq i \leq p$, has two vertices x and y such that $x, y \in W^{3'}$.*

4. *No $t_i, i \in \{1, p\}$, has three vertices $x, y,$ and z such that $x, y \in W^3$ and if $e_0 = x + y$ or $e_p = x + y$ then $z \in W^1 \cup W^2 \cup W^{3'}$.*

The 2-tree in Figure 3.2 is not an ssi2-lobster, since t_1 has two vertices in $W - W^3$, w_1 and w_2. The graphs A, B, and C in Figure 3.3 are sparse spiny interior 2-lobsters. In graph A, $\partial_1^1 A = \{j\}$ and $\partial_1^2 A = \{a, m, h\}$; thus $W^1 = W^3 = \{d\}$ and $W^2 = W^{3'} = \emptyset$. Because there is only one vertex in W, A is an ssi2-lobster. In graph B, $\partial_1^1 G = \{v\}$ and $\partial_1^2 B = \{a, b, r, t, n\}$; thus $W^1 = \{h\}$, $W^2 = \{h\}$, $W^3 = \{d, f, h\}$, and $W^{3'} = \emptyset$. Although t_1 has two vertices in W, neither of them are in $W^1 \cup W^2 \cup W^{3'}$. No other clique has more than one vertex in W, so B is an ssi2-lobster. In graph C, $\partial_1^1 C = \emptyset$ and $\partial_1^2 C =$

$\{a, f, g, j, n, q\}$. Both W^1 and W^2 are empty and $W^3 = \{d, c, k\}$. Although

there are two vertices f and g such that $N_G(f) = N_{G^{2-}}(b) + b - c = e_0 + b - c$

and $N_G(g) = N_{G^{2-}}(b) + b - c = e_0 + b - c$, because both are adjacent to b,

the vertex $c \notin W^{3'}$; hence $W^{3'} = \{d\}$. Because $W^{3'}$ contains only one vertex,

the third condition on being an ssi2-lobster must hold. Since t_1 contains three

vertices in W^3, we check the fourth condition and notice that $e_0 = d + c$ and

$e_p = d + k$ and in neither case is the third vertex in $W^1 \cup W^2 \cup W^{3'}$. Thus C is

also an ssi2-lobster. In order to prove that ssi2-lobsters are exactly the class of

2-trees that have probe interval representations, first we need some definitions

and lemmas in [47] and [43].
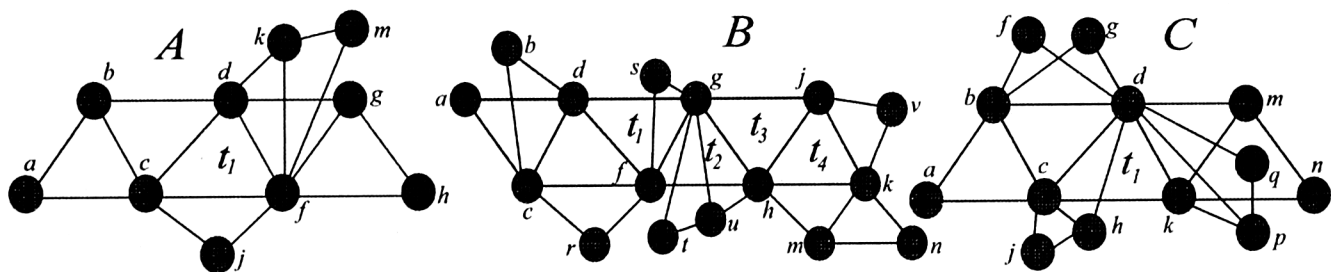
Figure 3.3: Three sparse spiny interior 2-lobsters with the vertices and 3-cliques of G^{2-} labeled.

A collection of sets $\{X, Y, Z\}$ is an *asteroidal collection* (AC) if for all $x \in X$,

for all $y \in Y$, and for all $z \in Z$, $\{x, y, z\}$ is an asteroidal triple. Each of these

sets X, Y, and Z is called as *asteroidal set* (AS). The following results in [43]

and [47] provide information about the (P, N)-partition of vertices in a PIG.

Lemma 3.2 *[47] In a PIG at least one vertex of every asteroidal triple must be*

a non-probe.

Corollary 3.3 *[43] At least one asteroidal set of an asteroidal collection of a*

PIG G must consist entirely of non-probes. Thus at least one asteroidal set of a PIG must be an independent set.

Lemma 3.4 *[43] In every asteroidal triple of a PIG G there must exist a non-probe asteroidal triple vertex u such that there exists a path between the other two asteroidal triple vertices that avoids $N(u)$ and has a non-probe internal vertex.*

Corollary 3.5 *[43] There exists only one (P, N)-partition of vertices of a 3-sun up to isomorphism.*

Figure 3.8 gives the (P, N)-partition of the 3-sun from Corollary 3.5. We now prove a lemma that says that the vertices of W^1 must be non-probes.

Lemma 3.6 *Let G be a spiny interior 2-lobster with a probe interval representation such that $G^{2-} = (e_0, t_1, e_1, t_2, \ldots, t_p, e_p)$. Any $w_x \in W^1$ must be a non-probe.*

Proof: Let G be a spiny interior 2-lobster with a probe interval representation such that $G^{2-} = (e_0, t_1, e_1, t_2, \ldots, t_p, e_p)$, and let $x \in \partial_1^1 G$ such that $N_G(x) = t_i - w_x$. If $i > 2$ then let $a \in t_{i-2} - e_{i-2}$, and let $b \in t_{i-1} - e_{i-1}$. If $i = 2$ let a be a 2-leaf of G^{1-} such that $N_{G^{1-}}(a) = e_0$, and let $b \in t_1 - e_1$. If $i = 1$ then let b be a k-leaf of G^{1-} such that $N_{G^{1-}}(b) = e_0$, and let $a \in \partial_1^2 G$ such that $N_G(a) \subset (e_0 + b)$. If $i < p - 1$ let $c \in t_{i+1} - e_i$ and let $d \in t_{i+2} - e_{i+1}$. If $i = p - 1$ then let d be a 2-leaf of G^{1-} such that $N_{G^{1-}}(d) = e_p$, and let $c \in t_p - e_{p-1}$. If $i = p$ then let c be a 2-leaf of G^{1-} such that $N_{G^{1-}}(c) = e_p$, and let $d \in \partial_1^2 G$ such that $N_G(d) \subset (e_p + c)$. Let $r \in t_i - e_i$. Let $s \in t_i - e_{i-1}$. Let $X = \{x\}$, $Y = \{a, b\}$, and $Z = \{c, d\}$. We claim that $\{X, Y, Z\}$ is an AC. The path abw_xcd

avoids $N_G(x)$. The path $abrx$ avoids $N_G(c)$ and $N_G(d)$. The path $dcsx$ avoids $N_G(a)$ and $N_G(b)$. By Corollary 3.3, since $ab \in E(G)$ and $cd \in E(G)$, the AS $X = \{x\}$ must contain all non-probes. Now consider the subgraph H induced by the vertices $\{b, c, r, s, x, w_x\}$, which is isomorphic to the 3-sun. By Corollary 3.5, since x must be a non-probe, w_x must also be a non-probe. ∎
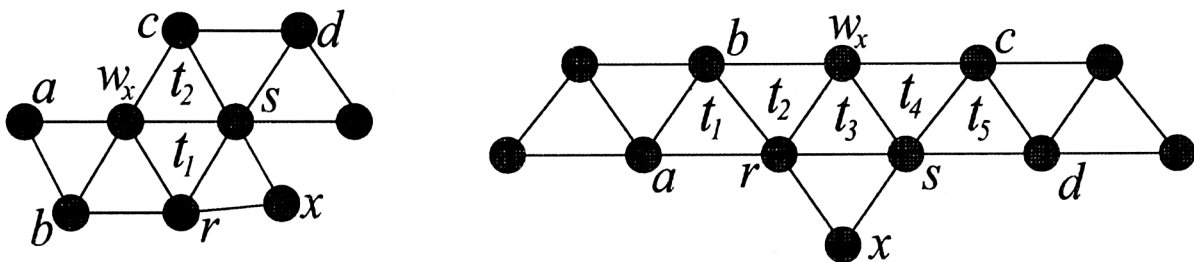
Figure 3.4: Two labeled examples for Lemma 3.6.

Figure 3.4 gives two ssi2-lobsters that are labeled as in the proof of Lemma 3.6. In order to prove a similar statement about vertices of W^2, we must first provide a lemma about the probes and non-probes of an induced subgraph that is common in 2-trees.

Lemma 3.7 *There are exactly three (P, N)-partitions of vertices of the PIG Q in Figure 3.8 up to isomorphism.*

Proof: Label the vertices of the PIG Q as in the Figure 3.8. The vertices $\{a, g, h\}$ form an AT, since the path acg avoids $N_Q(h)$, the path $abdh$ avoids $N_Q(g)$, and the path $hdfg$ avoids $N_Q(a)$. By Lemma 3.4, we know there must exist a non-probe AT vertex u such that there exists a path between the other two AT vertices that avoids $N(u)$ and has a non-probe internal vertex. First let us assume that h is the AT vertex with this property. The only vertex in the

neighborhood of h is d, so the internal vertex that must also be a non-probe can be either b, f, or c.

Assume that b is the non-probe vertex, and thus a and c must be probes. In a probe interval representation of G, I_h and I_c must not overlap, since they are not adjacent and $c \in P$. Furthermore, d is adjacent to both h and c. Thus without loss of generality, let $r_h < l_c$ and hence $l_d < r_h$ and $r_d > l_c$. Now both b and f are adjacent to d but not adjacent to one another or to h. Let us assume that f is a probe, so I_f and I_b must not overlap. If $I_f \subset I_d$, then $I_g \cap I_d \neq \emptyset$, which is a contradiction since d is a probe and d and g are not adjacent. Thus since f is not adjacent to h, we must have that $r_b < l_f$. However, this forces $I_a \cap I_d \neq \emptyset$, since a is adjacent to both c and b, which is a contradiction (see Figure 3.5).
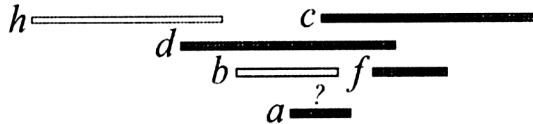
Figure 3.5: An example for the proof of Lemma 3.7 where any placement of the interval for a produces a contradiction.

Let us assume then that f is a non-probe and that $r_b < r_f$. Since a and g are adjacent to c and not to d, it must be that $r_d < l_a$ and $r_d < l_g$. Thus we have that $I_a \cap I_f \neq \emptyset$, which is a contradiction (see Figure 3.6). We get a similar contradiction if f is a non-probe and $r_f < r_b$, so if $h \in N$ then $b \notin N$.
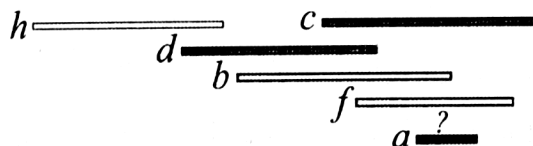
Figure 3.6: An example for the proof of Lemma 3.7 where any placement of the interval for a produces a contradiction.

We get similar arguments and contradictions if we assume that h and f must be the non-probes that satisfy Lemma 3.4, so it is also true that if $h \in N$, then $f \notin N$. Hence it must be the case that $N = \{h, c\}$ and $P = \{a, b, d, f, g\}$ (see Q' in Figure 3.8). With this (P, N)-partition and the interval assignment $I_c = (0, 6), I_a = (0, 2), I_b = (1, 2.5), I_d = (2, 4), I_f = (3.5, 5), I_g = (4, 6), I_h = (2.5, 3.5)$, we can see that this is a probe interval representation.

Now let us start with the AT non-probe vertex being a, which forces $b, c \in P$. By Lemma 3.4, we know that either d or f must be a non-probe. Assume that f is a non-probe, which means that d and g are probes. We also know from above that h must be a probe if f is a non-probe, and thus the only vertices that are non-probes are a and f. Since $N_Q(a) \nsubseteq N_Q(f)$ and $N_Q(f) \nsubseteq N_Q(a)$, neither interval can be contained in the other. We know that I_a and I_f must overlap, or else the probe interval representation would be an interval representation, which is a contradiction since G contains an AT. Without loss of generality, let $l_a < l_f$. Since $b \in N_Q(a)$, but $b \notin N_Q(f)$, we know that $r_b < l_f$. However, d is adjacent to both b and f, so $l_d < r_b$ and $r_d > l_f$. This forces $I_a \cap I_d \neq \emptyset$, which is a contradiction (see Figure 3.7). Therefore, d must be the non-probe. Hence either $N = \{a, d\}, P = \{b, c, f, h, g\}$ or $N = \{a, d, g\}, P = \{b, c, f, h\}$ (see Q in Figure 3.8). With either (P, N)-partition and the interval assignment $I_c = (2, 6), I_a = (2, 3), I_b = (2, 3), I_d = (1, 4), I_f = (3, 5), I_g = (4, 6), I_h = (0, 2)$, we can see that both are probe interval representations.

If we start by assuming that the AT non-probe vertex is g, we find a similar argument to the one above. However, these (P, N)-partitions are isomorphic to the ones above. Thus there are exactly three (P, N)-partitions of vertices of PIG
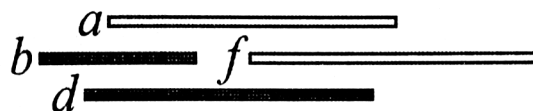
Figure 3.7: An example for the proof of Lemma 3.7, where the placement of the interval for d forces an adjacency between a and d, which is a contradiction.

Q in Figure 3.8 up to isomorphism.
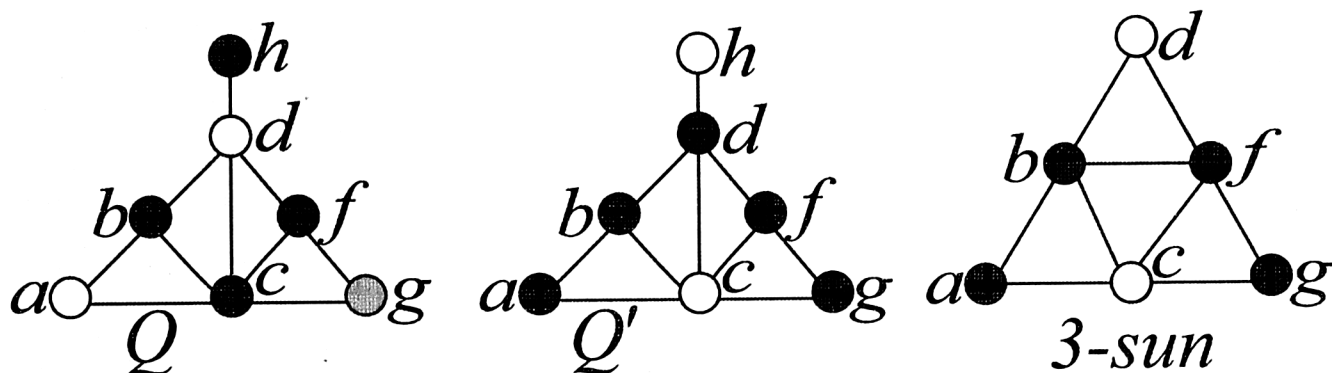
Figure 3.8: Graphs Q and the 3-sun with their possible (P, N)-partitions up to isomorphism. The white vertices are non-probes, black are probes, and the grey can be either a probe or non-probe.

Now we use Lemma 3.7 to prove a lemma that says that the vertices of W^2 must be non-probes.

Lemma 3.8 *Let G be a spiny interior 2-lobster with a probe interval representation such that $G^{2-} = (e_0, t_1, e_1, t_2, \ldots, t_p, e_p)$. Any $w_y \in W^2$ must be a non-probe.*

Proof: Let G be a spiny interior 2-lobster with a probe interval representation such that $G^{2-} = (e_0, t_1, e_1, t_2, \ldots, t_p, e_p)$. Let $y \in \partial_1^2$ such that $N_{G^{2-}}(y) = e_i - w_y$ for $w_y \in W^2(G)$; from the definition of W^2, $i \neq 0, p$. If $i > 2$ let $a \in t_{i-2} - e_{i-2}$ and let $b \in t_{i-1} - e_{i-1}$. If $i = 2$ let a be a 2-leaf of G^{1-} such that $N_{G^{1-}}(a) = e_0$, and let $b \in t_1 - e_1$. If $i = 1$ let b be a 2-leaf of G^{1-} such

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that $N_{G^{1-}}(b) = e_0$, and let $a \in \partial_1^2 G$ such that $N_G(a) \subset (e_0 + b)$. If $i < p - 2$ let $c \in t_{i+2} - e_{i+1}$, and let $d \in t_{i+3} - e_{i+2}$. If $i = p - 2$ let d be a 2-leaf of G^{1-} such that $N_{G^{1-}}(d) = e_p$, and let $c \in t_p - e_{p-1}$. If $i = p - 1$ let c be a 2-leaf of G^{1-} such that $N_{G^{1-}}(c) = e_p$, and let $d \in \partial_1^2 G$ such that $N_G(d) \subset (e_p + c)$. Let $s \in t_i - e_i$. Let $r \in t_{i+1} - e_i$. Lastly, let $N_G(y) = e_i + u - w_y$. This leads us to three cases. Either both $b, c \notin N_G(w_y)$, exactly one of b or $c \notin N_G(w_y)$, or $b, c \in N_G(w_y)$.

Case 1: The vertices $b, c \notin N_G(w_y)$.

If both $b, c \notin N_G(w_y)$, then the graph induced by vertices $\{b, s, w_y, r, c, u, y\}$ is a NL7. By [47] we know that the vertex of degree 3 in a NL7 must be a non-probe. Therefore, w_y is a non-probe.

Case 2: Exactly one of b or $c \in N_G(w_y)$.

Without loss of generality, assume that $b \in N_G(w_y)$ and $c \notin N_G(w_y)$. Label vertex v such that $v \in t_i$, but $v \notin e_{i-1}$, which implies that $v \in e_i$. Let $Y = \{y\}$, $X = \{a, b\}$, and $Z = \{c, d\}$, and we claim that $\{X, Y, Z\}$ is an AC. The path $abw_y rcd$ avoids $N_G(y)$; $abw_y uy$ avoids $N_G(c)$ and $N_G(d)$; and $dcvy$ avoids $N_G(b)$ and $N_G(a)$. Since $ab \in E(G)$ and $cd \in E(G)$, the AS $Y = \{y\}$ must contain all non-probes by Corollary 3.3. We know that $y \notin N_G(r)$ since $r \notin e_i$. We know that $u, y, r, c, v \notin N_G(b)$ because $b \notin e_{i-1}$. Thus the graph H induced by the vertices $\{b, w_y, r, c, v, u, y\}$ is isomorphic to Q in Figure 3.8, with $w_y \cong d$ and $y \cong a$. By Lemma 3.7 and the fact that $y \in N$, w_y is a non-probe.

Case 3: The vertices $b, c \in N_G(w_y)$.

Let $e_i = w_y + f$, and let $Y = \{y\}$, $X = \{a, b\}$, and $Z = \{c, d\}$; we claim $\{X, Y, Z\}$ is an AC. The path $abw_y cd$ avoids $N_G(y)$; $absfy$ avoids $N_G(c)$ and $N_G(d)$; and $dcrfy$ avoids $N_G(a)$ and $N_G(b)$. Since $ab \in E(G)$ and $cd \in E(G)$,

the AS $Y = \{y\}$ must contain all non-probes by Corollary 3.3. Now consider the subgraph H induced by the vertices $\{b, s, f, y, w_y, r, c\}$, which is isomorphic to Q' in Figure 3.8 with $w_y \cong c$ and $y \cong h$. By Lemma 3.7 and the fact that $y \in N$, the vertex w_y is a non-probe. ∎
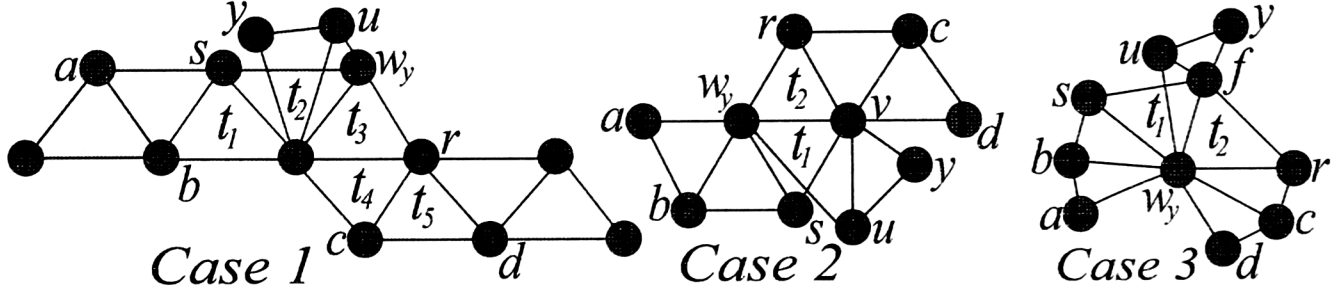
Figure 3.9: Examples for Cases 1, 2, and 3 in the proof of Lemma 3.8

We now find a new forbidden subgraph family not given in [43]. Although we will speak more of forbidden subgraphs in the next section, we prove that this family does not have a probe interval representation now so we can use it in the proof of the main theorem.

Lemma 3.9 *The family of graphs H_i, $i \in \mathbb{Z}^+$ are not probe interval graphs.*

Proof: Label H_i, $i \in \mathbb{Z}^+$ as in Figure 3.10. The subgraph induced by the vertices $\{w_y, w_x, r, g, y, f, h\}$ is an NL7, which forces w_y to be a non-probe. Now consider subgraph induced by the vertices $\{r, b, x, d_1, w_x, w_y\}$, which is a 3-sun with w_y as a non-probe. By Corollary 3.5, b must also be a non-probe. Lastly, let $X = \{x\}$, $Y = \{r\}$, and $Z = \{g, y\}$. We claim that $\{X, Y, Z\}$ is an AC. The path $r w_x w_y g y$ avoids $N_G(x)$, the path rbx avoids $N_G(g)$ and $N_G(y)$, and the path $x d_1 w_y g y$ avoids $N_G(r)$. By Lemma 3.3 and the fact that g and y are adjacent, either r or x must be a non-probe. However, we already said

that b must be a non-probe, and both r and x are adjacent to b, which is a contradiction. Therefore, no H_i, $i \in \mathbb{Z}^+$ is a probe interval graph. ∎
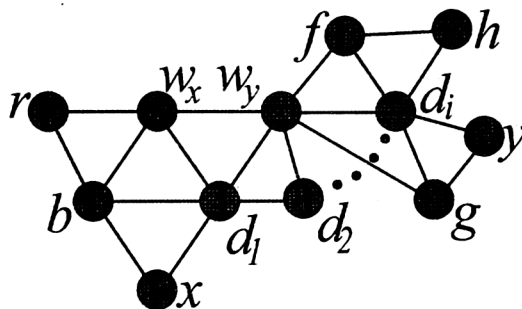
Figure 3.10: H_i, $i \in \mathbb{Z}^+$

Corollary 3.10 *The graph S in Figure 3.15 is not a PIG.*

Proof: The only vertices used in the proof that $H_i, i \in \mathbb{Z}^+$ are not PIGs are in S. Therefore, S is not a PIG. ∎

3.2 Probe Interval 2-tree Characterization

Now that we have established much of the structure of a probe interval 2-tree, we proceed with the main result of this chapter.

Lemma 3.11 *Let G be a 2-tree. If G is an ssi2-lobster, then G is a PIG.*

Proof: Suppose G is a sparse spiny interior 2-lobster such that $G^{2-} = (e_0, t_1, e_1, ..., t_p, e_p)$. By the definition of e_0 and e_p, we know that both contain at least one vertex in W^3. If e_0 contains exactly one vertex in W^3, label it w. Similarly if e_p contains exactly one vertex in W^3, then label it a. If both vertices of e_0(or e_p) are in $W^3 - (W^1 \cup W^2 \cup W^{3'})$, then label the vertex in e_0 and e_1 as w (or e_p and e_{p-1} as a). If not then by conditions $1, 2$, and 3 of the definition of an ssi2-lobster, both vertices of e_0 (and e_p) are not in the set $W^1 \cup W^2 \cup W^{3'}$.

Thus choose a vertex $w \in e_0$ such that $w \in W^3 - (W^1 \cup W^2 \cup W^{3'})$, and choose a vertex $a \in e_p$ such that $a \in W^3 - (W^1 \cup W^2 \cup W^{3'})$. Let $y \in \partial_1^2$ such that $N_G(y) = N_{G^1-}(z) + z - w = e_0 + z - w$. Label $e_0 = w + x$, and notice that $N_{G^1-}(z) = \{w, x\}$ and $y \in N_G(x)$. Similarly let $d \in \partial_1^2$ such that $N_G(d) = N_{G^1-}(c) + c - a = e_p + c - a$, and label $e_p = a + b$. Now we let $e_{-2} = y + z$, $t_{-1} = x + y + z$, $e_{-1} = z + x$, $t_0 = z + w + x$, $t_{p+1} = a + b + c$, $e_{p+1} = b + c$, $t_{p+2} = b + c + d$, and $e_{p+2} = c + d$.

Make each vertex $v \in W^1 \cup W^2 \cup W^{3'} \cup (W^3 - \{w, a\})$ and each vertex $z \in \partial_1 - \{y, d\}$ a non-probe. If w or a is in $W^1 \cup W^2 \cup W^{3'}$ then it will be a non-probe; however, if it is just in $W^3 - W^{3'}$ then it will be a probe. By conditions $1, 2, 3$ of the definition of an ssi2-lobster, no clique contains two vertices from $W^1 \cup W^2 \cup W^{3'}$, and from condition 4 if a clique contains a vertex from $(W^3 - \{w, a\})$ then it does not contain a vertex from $W^1 \cup W^2 \cup W^{3'}$. Hence no clique in G^{2-} contains two non-probe vertices; therefore from the definition of a 2-tree, no two non-probe vertices of G^{2-} are adjacent. Furthermore, no two vertices of ∂_1 are adjacent, and if a vertex of ∂_1 is adjacent to a vertex of $W^1 \cup W^2 \cup W^{3'} \cup (W^3 - \{w, a\})$ then there is a clique with two vertices in $W^1 \cup W^2 \cup W^{3'}$ and G is not an ssi2-lobster. Thus this (P, N)-partition has no adjacencies between non-probe vertices. Now we must assign intervals using this (P, N)-partition.

Let $G^{2-*} = (e_{-2}, t_{-1}, e_{-1}, t_0, e_0, t_1, e_1, ..., t_p, e_p, t_{p+1}, e_{p+1}, t_{p+2}, e_{p+2})$, and for each $v \in G^{2-*}$ assign an ordered pair (m, n) such that t_m is the first clique that contains v and t_n is the last. Assign the interval $I_{v_{m,n}} = (m, n + \frac{1}{2})$ to each $v \in G^{2-*}$. Notice that the interval $(i + \frac{1}{2}, i + 1)$ contains only the vertices from e_i,

and the interval $(i, i+\frac{1}{2})$ contains only the vertices from t_i. For each $e_i \in G^{2-*}$, let

$M_i = \{v \in V(G) : v \notin V(G^{2-*})$ and $N_{G^{2-*}}(v) = e_i\}$ and enumerate the vertices

of M_i as $x_{(i,1)}$ to $x_{(i,|M_i|)}$. Assign the interval $I_{x_{(i,j)}} = (i+\frac{1}{2}+\frac{j-1}{2|M_i|}, i+\frac{1}{2}+\frac{j}{2|M_i|})$ to

each $x_{(i,j)} \in M_i$ for all M_i. For each $x_{(i,j)} \in M_i$ let $M_{i,j} = \{v \in \partial_1^2 : x_{(i,j)} \in N(v)\}$

and enumerate the vertices of $M_{(i,j)}$ as $y_{(i,j,1)}$ to $y_{(i,j,|M_{(i,j)}|)}$. Assign the interval

$I_{y_{(i,j,k)}} = (i+\frac{1}{2}+\frac{j-1}{2|M_i|}, i+\frac{1}{2}+\frac{j}{2|M_i|})$ to each $y_{(i,j,k)} \in M_{(i,j)}$. Recall that all

these vertices are non-probes, so their intersecting intervals do not represent

adjacencies. Since the spiny interior 2-lobster is sparse, each $y_{(i,j,k)} \in M_{(i,j)}$ in

not adjacent to a distinct vertex of e_i, which is a non-probe. Therefore, no

adjacency results between each $y_{(i,j,k)}$ and its non-adjacency in e_i.

For each $t_i, 1 \le i \le p$ let $Q_i = \{v \in \partial_1^1 : N_G(v) \subset t_i\}$, and let $Q_0 = \{v \in \partial_1^2 :$

$N_G(v) = z + w\}$ and $Q_{p+1} = \{v \in \partial_1^2 : N_G(v) = a + c\}$. Enumerate the vertices

of $Q_i, 0 \le i \le p+1$ as $c_{(i,1)}$ to $c_{(i,|Q_i|)}$. Assign the interval $I_{c_{(i,j)}} = (i, i+\frac{1}{2})$.

Recall that all these vertices are non-probes, so their intersecting intervals do

not result in an adjacency. Since G is an ssi2-lobster, the vertex $w_i \in t_i$ such

that $w_i \notin N_{G^{2-*}}(c_{(i,j)})$ is the same for all j, and w_i is a non-probe since it is in

W^1. Therefore no adjacency results between each $c_{(i,j)}$ and w_i for all j. Thus G

has a probe interval representation. ∎

Figure 3.11 gives an example of an ssi2-lobster with the vertices labeled as

in Lemma 3.11. Both x and w are in W^3 and neither is in $W^1 \cup W^2 \cup W^{3'}$, thus

the vertex in both e_0 and e_1 was chosen as w. This is because the other vertex

x is then forced to be a non-probe, even though it is not in $W^1 \cup W^2 \cup W^{3'}$.

Since it is only adjacent to w on G^{2-} and w is a probe, we avoid two non-probe

vertices in the same clique. Given w's labeling, $x_{(-1,1)}$ could have been labeled
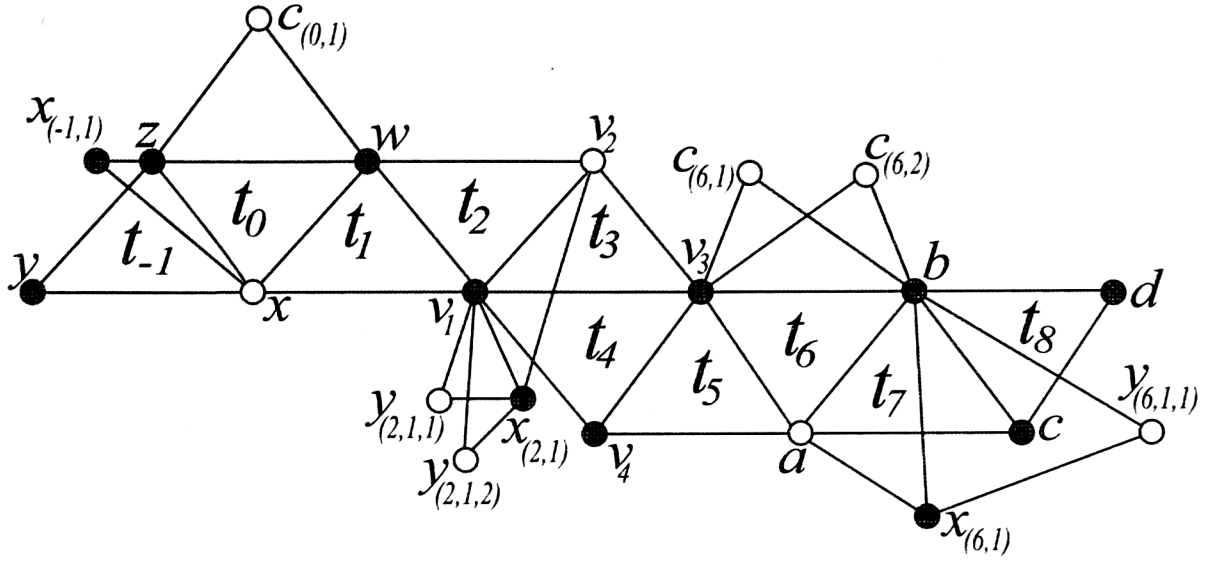
Figure 3.11: A ssi2-lobster with vertices labeled as in Lemma 3.11

as y. Notice on the other end of the 2-tree that a is the only vertex in e_6 that is in W^3, thus it was labeled as such. Again the choice of the vertex that was labeled d was arbitrary.

Figure 3.12 gives the probe interval assignment for Figure 3.11. The representation has each clique t_i from G^{2-*} cover the interval $(i, i+\frac{1}{2})$. In this section we can add intervals for 2-leaves of G in ∂_1^1 whose neighborhood is a subset of t_i. Since there is at most one vertex which is not adjacent to these vertices, it becomes a non-probe. Likewise, the representation has each clique e_i covering the interval $(i+\frac{1}{2}, i+1)$. We can partition this interval for each vertex that is adjacent to e_i but is not on G^{2-*}. Next we add intervals for vertices that are 2-leaves of G and whose neighborhood is a subset of the vertices in this area. However, we must make the 2-leaf and its non-adjacency on G^{2-} a non-probe. This is the intuition for the ssi2-lobster: you can't have two of these non-adjacencies for 2-leaves in one clique, because they must be non-probes (although we have

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more flexibility at the ends of the 2-path G^{2-}).
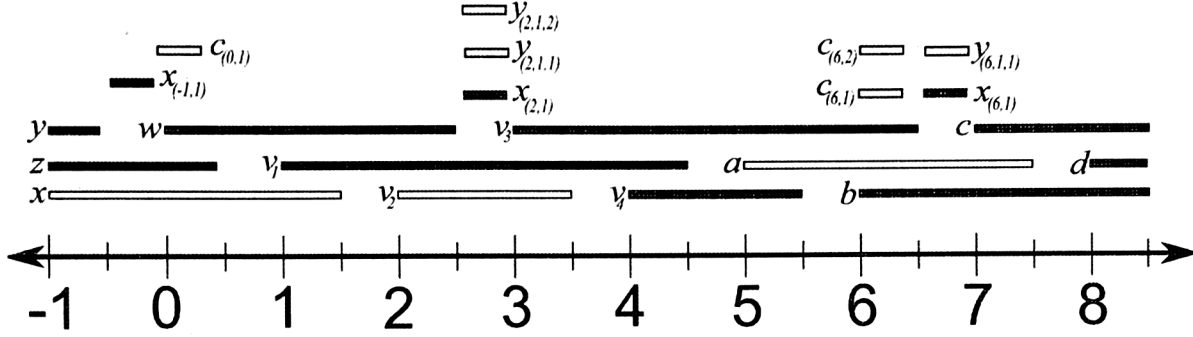
Figure 3.12: The probe interval representation for Figure 3.11

Theorem 3.12 *Let G be a 2-tree. G is a probe interval graph if and only if it is a ssi2-lobster.*

Proof: Let G be a 2-tree that has a probe interval representation. Assume that it is not a *ssi*2-lobster. Thus it is either not a spiny interior 2-lobster or it is a spiny interior 2-lobster that is not sparse. If it is not a spiny interior 2-lobster, then by Theorem 2.7 it is not an interval p-graph. Since probe interval graphs are contained in interval p-graphs, it is not a probe interval graph.

Therefore, we assume that G is a spiny interior 2-lobster that is not sparse. Let $G^{2-} = (e_0, t_1, e_1, ..., t_p, e_p)$. Either there is a t_i with two vertices w_x and w_y in $W - W^3$, there is a t_i with two vertices w_x and w_y such that $w_x \in W - W^3$ and $w_y \in W^{3'}$, there is a t_i with two vertices w_x and w_y such that $w_x, w_y \in W^{3'}$, or there is a $t_i, i \in \{1, p\}$, with three vertices w_x, w_y, and w_z such that $w_x, w_z \in W^3$ and if $e_0 = w_x + w_z$ or $e_p = w_x + w_z$ then $w_y \in W^1 \cup W^2 \cup W^{3'}$.

Case 1: There is a t_i with two vertices w_x and w_y in $W - W^3$.

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In this case, w_x and w_y are non-probes by Lemmas 3.6 and 3.8, which is a contradiction since $w_x w_y \in E(G)$.

Case 2: There is a t_i with two vertices w_x and w_y such that $w_x \in W - W^3$ and $w_y \in W^{3'}$

Without loss of generality, let $N_G(y) = N_{G^2-}(z) + z - w_y = e_p + z - w_y$ and $N_G(s) = N_{G^2-}(r) + r - w_y = e_p + r - w_y$ for $y, s \in \partial_1^2 G$ and distinct $z, r \in V(G)$. Since w_x and w_y are both in t_i, $w_y \in e_j$, $i \le j \le p$. If $w_y \notin e_{i-1}$, there is a vertex g such that $e_{i-1} \subset N_G(g)$ and $w_y \notin N_G(g)$. Let $e_{i-1} = w_x + h$. The subgraph H induced by the vertices $\{w_y, h, g, y, s, z, r\}$ is an NL7 with w_y the vertex of degree 3 and hence is forced to be a non-probe (see A_1 in Figure 3.13). By Lemmas 3.6 and 3.8, w_x is a non-probe, which is a contradiction since $w_x w_y \in E(G)$.

Therefore, let us assume that $w_y \in e_{i-1}$. Since w_y is in both e_{i-1} and e_i, this precludes w_x from being in W^1 since vertices in this set must be in two consecutive e_j sets. This means that $w_x + w_y = e_i$ for $i \ne 0$ and $i \ne p$. Let $e_p = w_x + f$ and $b \in e_{i-1}$ and consider the subgraph H induced be the vertices $\{w_y, f, z, y, r, s, b\}$, which is isomorphic to Q in Figure 3.8 with $w_y \cong d$ (see A_2 in Figure 3.13). Since $w_x b \in E(G)$, b must be a probe. Thus by Lemma 3.7, w_y must be a non-probe and we again get a contradiction.

Case 3: There is a t_i with two vertices w_x and w_y such that $w_x, w_y \in W^{3'}$.

Let $N_G(y) = N_{G^2-}(z) + z - w_y = e_i + z - w_y$ and $N_G(s) = N_{G^2-}(r) + r - w_y = e_i + r - w_y$ for $y, s \in \partial_1^2 G$ and distinct $z, r \in V(G)$, ($i = 0$ or $i = p$), and $N_G(x) = N_{G^2-}(v) + v - w_x = e_i + v - w_x$ and $N_G(c) = N_{G^2-}(d) + d - w_x = e_i + d - w_x$ for $x, c \in \partial_1^2 G$ and distinct $v, d \in V(G)$, ($i = 0$ or $i = p$). First consider the case that $w_x + w_y \ne e_0$ nor e_p. Then the subgraph H induced by the
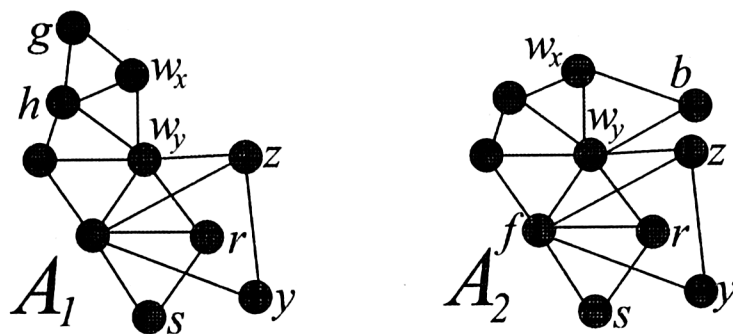
Figure 3.13: Labeled examples for Case 2 in the proof of Theorem 3.12.

vertices $\{w_y, y, z, s, r, w_x, v\}$ is isomorphic to an NL7 with w_y the vertex of degree 3. Similarly, the subgraph H induced by the vertices $\{w_x, v, d, x, c, w_y, z\}$ is isomorphic to an NL7 with w_x the vertex of degree 3 (see A_3 in Figure 3.14). These two induced subgraphs force both w_x and w_y to be non-probes, which is a contradiction since they are adjacent.

Therefore, let us assume without loss of generality that $w_x + w_y = e_p$ and that $w_x \notin e_{p-1}$. Let $a \in t_p$ such that $a \neq w_x, w_y$, and notice that there exists a vertex b such that $a, w_y \in N_G(b)$, but $b \notin N_G(w_x)$. Thus the subgraph H induced by the vertices $\{w_x, v, x, c, d, a, b\}$ is an NL7 with w_x the vertex of degree 3 and hence must be a non-probe. If $N_{G^{1-}}(z) = N_{G^{1-}}(r) = e_0$, then $e_0 = w_y + f$ and the subgraph H induced by the vertices $\{w_y, z, y, r, s, d, f\}$ is isomorphic to Q in Figure 3.8 (see A_4 in Figure 3.14). Since $d \in N_G(w_x)$ and w_x is a non-probe, d must be a probe. By Lemma 3.7 this forces w_y to be a non-probe, which is again a contradiction.

If $N_{G^{1-}}(z) = N_{G^{1-}}(r) = e_p$, then let g be a vertex such that $g \notin t_p$ and $g \in N_G(b)$. We know such a vertex exists since G^{2-} has at least one t_i. The subgraph H induced by the vertices $\{w_y, y, z, s, r, w_x, b\}$ is isomorphic to Q in

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Figure 3.8. By Lemma 3.7 and the fact that $w_x \in N$, w_x and b must be non-probes. If $g \in N_G(w_y)$, consider the subgraph H induced by the vertices $\{w_y, y, z, s, r, w_x, g\}$, which is again isomorphic to Q in Figure 3.8. Hence by Lemma 3.7, w_x and g must be non-probes, but this is a contradiction since $bg \in E(G)$. If $g \notin N_G(w_y)$ consider the subgraph H induced by the vertices $\{w_y, y, z, s, r, w_x, b, g\}$. We claim the set $\{g, y, s\}$ is an AT (see A_5 in Figure 3.14). The path gbw_yrs avoids $N_G(y)$; the path gbw_yzy avoids $N_G(s)$; and the path sw_xy avoids $N_G(g)$. By Lemma 3.4, we know that one of the vertices in the AT must be a non-probe such that there exists a path between the other two AT-vertices that avoids the neighborhood of the non-probe and has a non-probe internal vertex. Since g is adjacent to b, which is a non-probe, the non-probe AT vertex must be either y or s. However, both of these vertices are adjacent to w_x, which has already been shown to be a non-probe, and thus we get a contradiction.
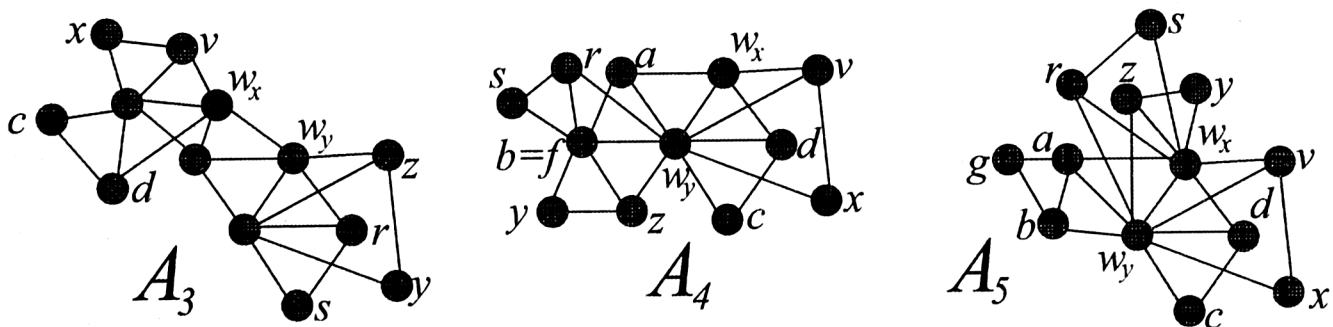
Figure 3.14: Labeled examples for Case 3 in the proof of Theorem 3.12.

Case 4: There is a t_i, $i \in \{1, p\}$, with three vertices w_x, w_y, and w_z such that $w_x, w_z \in W^3$ and if $e_0 = w_x + w_z$ or $e_p = w_x + w_z$ then $w_y \in W^1 \cup W^2 \cup W^{3'}$

Without loss of generality, let $e_p = w_x + w_z$, $e_{p-1} = w_x + w_y$ and $N_G(x) = N_{G^{2-}}(v) + v - w_x = e_p + v - w_x$ and $N_G(z) = N_{G^{2-}}(d) + d - w_z = e_p + d - w_z$ for $x, z \in \partial_1^2 G$ and $v, d \in V(G)$. We first consider the case that $v = d$. If $y \in W^{3'}$, then let $N_G(y) = N_{G^{2-}}(c) + c - w_y = e_0 + c - w_y$ and $N_G(s) = N_{G^{2-}}(r) + r - w_y = e_0 + r - w_y$ for $y, s \in \partial_1^2 G$ and distinct $c, r \in V(G)$. The subgraph induced by all the vertices of G^{2-} and $\{r, s, c, y, v, x, z\}$ is isomorphic to H_i in Figure 3.10 for some $i \in \mathcal{Z}^+$, and hence by Lemma 3.17 G is not a PIG.

Now let us assume that $w_y \in W^1 \cup W^2$, and recall that by Lemmas 3.6 and 3.8, w_y must be a non-probe. The graph H induced by the vertices $\{w_y, w_z, w_x, v, z, x\}$ is isomorphic to the 3-sun. With w_y already a non-probe, we know that v must also be a non-probe by Corollary 3.5. Since w_y is in $W^1 \cup W^2$, the clique t_{p-1} must exist on G^{2-}, so without loss of generality we let $t_{p-1} = a + w_y + w_x$. We claim that $\{a, z, x\}$ is an asteroidal triple. The path $a w_x z$ avoids $N_G(x)$; the path $a w_y w_z x$ avoids $N_G(z)$; and the path $z v x$ avoids $N_G(a)$. We know that one of the vertices in the AT must be a non-probe, but since a is adjacent to w_y, the non-probe must be either z or x. However, both z and x are adjacent to v, which has already been shown to be a non-probe, and we have a contradiction.

Now we consider the case that $v \neq d$. If $w_y \in W^{3'}$, then subgraph H induced by the vertices $\{w_y, c, y, r, s, w_z, x\}$ is isomorphic to NL7 with w_y the vertex of degree 3, hence w_y must be a non-probe. By Lemmas 3.6 and 3.8, if $w_y \in W^1 \cup W^2$, then w_y is a non-probe. In either case, there is a vertex r that is either a 2-leaf of G^{1-} or in t_{p-1} such that $w_x, w_y \in N_G(r)$. Now consider the subgraph H induced by the vertices $\{w_x, w_y, w_z, v, d, x, z, r\}$, which is isomorphic

to G_1 in Figure 3.15. The graph $G_1 - v$ is isomorphic to Q in Figure 3.8. By Lemma 3.7, w_y cannot be a non-probe, which is a contradiction.

Since in every case we get a contradiction, if G is not a sparse spiny interior 2-lobster, then it is not a PIG.
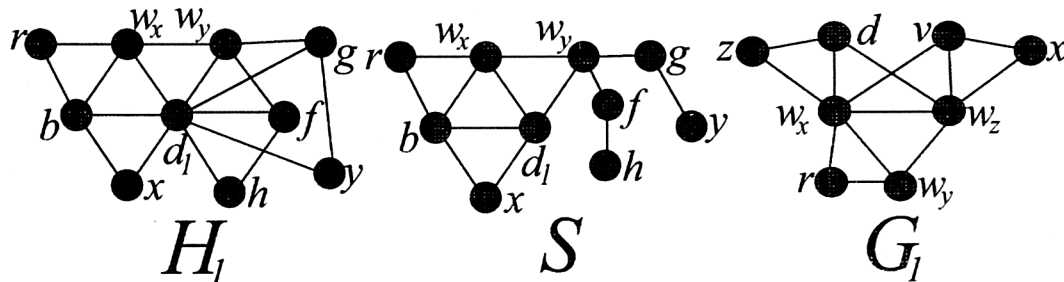
Figure 3.15: The graphs H_1, S, and G_1.

Now let G be a ssi2-lobster. By Lemma 3.11 G has a probe interval representation.

$\blacksquare$

3.3 Forbidden Subgraphs for Probe Interval 2-trees

Since this research was motivated in part by the 62 forbidden subgraphs found by Pržulj and Corneil in [43], we now turn our attention to forbidden subgraphs for probe interval 2-trees. We have already given one new family of forbidden subgraphs for 2-tree PIGS in Figure 3.10. If any vertex is deleted from the infinite family, the graph ceases to be a 2-tree or has a probe interval representation. However, only one of the the graphs is minimal.

Proposition 3.13 *The graph H_1 is a minimal forbidden induced subgraph for 2-tree PIGs.*

Proof: Label H_1 as in Figure 3.15. The removal of any 2-leaf, $\{r, x, h, y, \}$ results in a ssi2-lobster; hence by Theorem 3.12 it has a probe interval representation. The removal of w_x or w_y results in a graph with no cycles larger than 3 and no AT, so it is an interval graph, which is also a probe interval graph. The removal of g is isomorphic to the removal of f. The removal of $\{g, b, d_1\}$ has a probe interval representation as illustrated in Figure 3.16. ∎
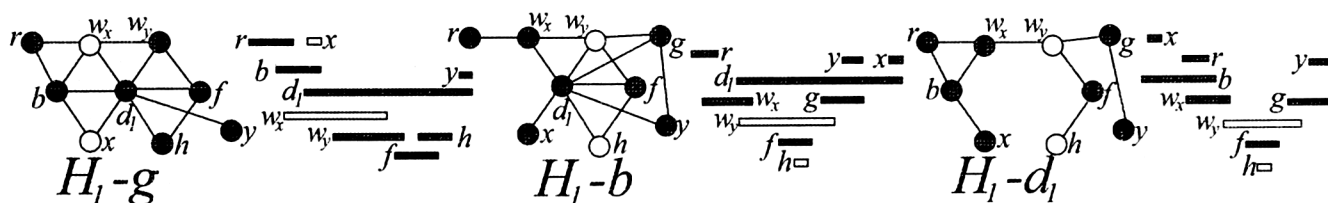
Figure 3.16: Three probe interval representations for H_1 minus a vertex.

The graphs $H_i, i \in \mathbb{Z}^+ - \{1\}$ are not minimal, since removing all d_i except d_1 results in the graph S in Figure 3.15, and by Corollary 3.10, S is not a PIG. However, there is no subgraph isomorphic to any $H_i, i \in \mathbb{Z}^+ - \{1\}$ in the 62 forbidden 2-trees in [43]. To see this, we have listed the degree sequences for the 62 forbidden subgraphs in the table below. It is easy to check that none of the $H_i, i \in \mathbb{Z}^+ - \{1\}$ have the same degree sequence as the graphs listed in the table below. For example, the degree sequences for the first four H_i are: $\{2, 2, 2, 2, 3, 3, 4, 4, 4, 8\}$, $\{2, 2, 2, 2, 3, 3, 4, 4, 5, 5, 6\}$, $\{2, 2, 2, 2, 3, 3, 3, 4, 4, 5, 6, 6\}$, and $\{2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 5, 6, 7\}$. Using the characterization of probe interval 2-trees being ssi2-lobsters, we can find more forbidden subgraphs that are not given in the original 62 in [43].

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S_1	$\{2,2,2,2,3,3,3,4,5,6\}$	S_{17}^4	$\{2,2,2,2,3,3,3,3,3,4,4,4,5,10\}$
S_2	$\{2,2,2,2,3,3,3,5,5,5\}$	S_{17}^5	$\{2,2,2,2,3,3,3,4,5,6\}$
S_3	$\{2,2,2,3,3,4,4,4,5,6\}$	S_{17}^6	$\{2,2,2,2,3,3,3,3,3,3,3,4,4,4,5,12\}$
S_4	$\{2,2,2,3,3,3,3,4,4,5,7\}$	S_{18}^1	$\{2,2,2,2,3,3,3,3,5,6,7\}$
S_5	$\{2,2,2,3,3,3,3,3,3,6,8\}$	S_{18}^2	$\{2,2,2,2,3,3,3,3,4,5,6,7\}$
S_6	$\{2,2,2,3,3,3,3,3,4,6,7\}$	S_{18}^3	$\{2,2,2,2,3,3,3,3,3,4,5,6,8\}$
S_7	$\{2,2,2,3,3,3,3,3,4,5,8\}$	S_{18}^4	$\{2,2,2,2,3,3,3,3,3,3,4,5,6,9\}$
S_8	$\{2,2,2,3,3,3,3,3,3,3,4,4,9\}$	S_{19}	$\{2,2,2,2,3,3,4,4,4,6,6\}$
S_{10}	$\{2,2,2,3,3,3,3,4,4,4,8\}$	S_{20}	$\{2,2,2,2,3,3,3,3,4,5,9\}$
S_{11}	$\{2,2,2,3,3,3,4,4,4,4,7\}$	S_{21}	$\{2,2,2,2,3,3,3,4,4,5,8\}$
S_{12}	$\{2,2,2,3,3,3,3,4,4,6,6\}$	S_{22}	$\{2,2,2,2,3,3,3,3,4,7,7\}$
S_{13}	$\{2,2,2,2,3,3,3,4,5,6,6\}$	S_{23}	$\{2,2,2,2,3,3,3,4,4,6,7\}$
S_{14}^1	$\{2,2,2,2,3,3,3,4,5,8\}$	S_{24}	$\{2,2,2,2,3,3,4,4,4,5,7\}$
S_{14}^2	$\{2,2,2,2,3,3,3,4,4,4,9\}$	S_{25}	$\{2,2,2,2,3,3,5,5,5,5\}$
S_{14}^3	$\{2,2,2,2,3,3,3,3,4,4,4,10\}$	S_{26}^1	$\{2,2,2,2,3,3,4,4,6,6\}$
S_{14}^4	$\{2,2,2,2,3,3,3,3,3,4,4,4,11\}$	S_{26}^2	$\{2,2,2,2,3,3,4,4,4,5,7\}$
S_{14}^5	$\{2,2,2,2,3,3,3,3,3,3,4,4,4,12\}$	S_{26}^3	$\{2,2,2,2,3,3,3,4,4,4,5,8\}$
S_{14}^6	$\{2,2,2,2,3,3,3,3,3,3,3,4,4,4,13\}$	S_{26}^4	$\{2,2,2,2,3,3,3,3,4,4,4,5,9\}$
S_{15}^1	$\{2,2,2,2,3,3,3,5,5,7\}$	S_{26}^5	$\{2,2,2,2,3,3,3,3,3,4,4,4,5,10\}$
S_{15}^2	$\{2,2,2,2,3,3,3,4,4,5,8\}$	S_{27}^1	$\{2,2,2,2,3,3,4,5,5,6\}$
S_{15}^3	$\{2,2,2,2,3,3,3,3,4,4,5,9\}$	S_{27}^2	$\{2,2,2,2,3,3,4,4,4,5,7\}$
S_{15}^4	$\{2,2,2,2,3,3,3,3,3,4,4,5,10\}$	S_{27}^3	$\{2,2,2,2,3,3,4,4,4,4,5,8\}$
S_{16}^1	$\{2,2,2,2,3,3,3,3,4,7,7\}$	S_{27}^4	$\{2,2,2,2,3,3,3,3,4,4,4,5,9\}$
S_{16}^2	$\{2,2,2,2,3,3,3,3,4,4,6,8\}$	S_{27}^5	$\{2,2,2,2,3,3,3,3,3,4,4,4,5,10\}$
S_{16}^3	$\{2,2,2,2,3,3,3,3,3,4,4,6,9\}$	S_{28}^1	$\{2,2,2,2,3,4,4,4,5,6\}$
S_{16}^4	$\{2,2,2,2,3,3,3,3,3,3,4,4,6,10\}$	S_{28}^2	$\{2,2,2,2,3,4,4,4,5,6\}$
S_{16}^5	$\{2,2,2,2,3,3,3,3,3,3,3,4,4,6,11\}$	S_{28}^3	$\{2,2,2,2,3,4,4,4,4,5,6\}$
S_{16}^6	$\{2,2,2,2,3,3,3,3,3,3,3,3,4,4,6,12\}$	S_{28}^4	$\{2,2,2,2,3,3,3,4,4,4,4,5,8\}$
S_{17}^1	$\{2,2,2,2,3,3,3,4,4,6,7\}$	S_{29}	$\{2,2,2,2,3,4,4,4,5,6\}$
S_{17}^2	$\{2,2,2,2,3,3,3,4,4,4,5,8\}$	$NL10$	$\{1,1,1,2,2,2,2,2,2,3\}$
S_{17}^3	$\{2,2,2,2,3,3,3,3,4,4,4,5,9\}$	$H10$	$\{1,1,1,1,2,2,2,2,3,3\}$

Proposition 3.14 *The graphs N_1 though N_6 are forbidden subgraphs for probe interval 2-tress that are not isomorphic to the any of the 62 graphs in [43].*

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Figure 3.17: The graphs N_1 through N_6 are not probe interval 2-trees.

Proof: In each 2-tree N_i, $1 \leq i \leq 6$, t_1 has two vertices in W^2. Thus they are not ssi2-lobsters and do not have a probe interval representations.

The degree sequences of N_i, $1 \leq i \leq 6$ are: $\{2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 7, 7\}$, $\{2, 2, 2, 2, 3, 3 , 3, 3, 3, 3, 6, 10\}$, $\{2, 2, 2, 2, 3, 3, 3, 3, 3, 4, 6, 9\}$, $\{2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 8, 8\}$, $\{2, 2, 2, 2, 3, 3, 3, 3, 3, 4, 7, 8\}$, $\{2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 6, 8\}$, respectively. It is easy to check using the table of the degree sequences of the 62 forbidden subgraphs in [43] above that the only degree sequences that are the same are those of N_6 and S_{16}^2. Let us assume for contradiction that these two 2-trees are isomorphic. Both of these graphs have exactly one vertex of degree 6, vertices a and z in Figure 3.18. Hence these vertices must map to one another in an isomorphism. However, the vertex $a \in V(S_{16}^2)$ has three neighbors of degree 3, vertices d, e, and f in Figure 3.18, and vertex $z \in V(N_6)$ only has two neighbors of degree 3, vertices x and u in Figure 3.18. This is a contradiction, so N_6 and S_{16}^2 are not isomorphic, and N_i, $1 \leq i \leq 6$, are not isomorphic to any of the 62 graphs in [43]. ∎

Although the graphs N_1 through N_6 have no induced subgraph isomorphic to an already known forbidden 2-tree PIG, they are not minimal in the traditional sense. For example, in Figure 3.19 $N_1 - x$ is not a PIG, so N_1 is not minimal.
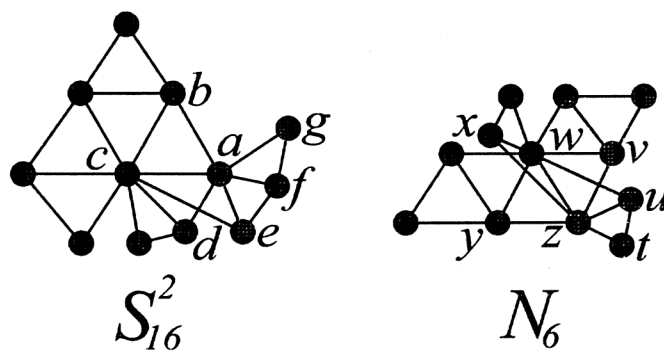
Figure 3.18: The graphs S_{16}^2 and N_6 are not isomorphic.

However, $N_1 - x$ is not a 2-tree or a tree, which brings us back to the discussion about minimality at the end of the last chapter. If you remove a vertex from the graphs N_1 through N_6, either they cease to be 2-trees or they have probe interval presentations. For example, in Figure 3.19 $N_1 - y$ is an ssi2-lobster, and thus has a probe interval representation.
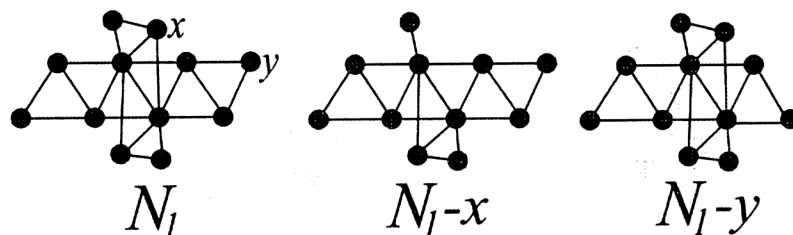
Figure 3.19: The graph N_1 and two induced subgraphs of N_1, one a PIG and one not a PIG.

In [43] S_3 through S_{12} are isomorphic to G_1 to G_{10} in the last chapter, and thus are not minimal. However, Pržulj and Corneil only eliminated S_9 from the count of 62 because it contains an NL10 (see Figure 1.1). They say their intent is to find forbidden 2-tree PIGs that do not contain forbidden tree PIGs as induced subgraphs, which is why they include H10 and NL10 in the count to 62. However, later in [43] infinite families of forbidden subgraphs are reduced to finite families by throwing out the graphs that are not minimal. Thus the

list of 62 is a combination of minimal and non-minimal forbidden subgraphs for probe interval 2-trees. To stay consistent, we add to their list H_1, since it is a minimal member of an infinite family, and N_1 to N_6 since it is a finite family. This brings us to our final proposition of this chapter.

Proposition 3.15 *There exist at least 69 graphs in the forbidden induced subgraph characterization for 2-tree PIGs.*

4. List-distinguishing Dihedral Graphs

In this chapter, we begin investigations into the list-distinguishing number of a graph. We start by giving a list-distinguishing Brooks-type theorem. Then we prove that $D_\ell(G) = D(G)$ for all graphs that realize the dihedral group. One interesting and useful consequence of this is that we determine precisely those graphs that realize D_n and have (traditional) distinguishing number 3.

4.1 Brooks-type Theorem

Theorem 4.1, known as Brooks' Theorem, gives a bound on the chromatic number of a graph based on the maximum degree of the graph. In this section we give a similar bound for the list-distinguishing number of a graph also based on the maximum degree of the graph.

Theorem 4.1 ([10]) *If G is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.*

Brooks-type theorems for the (traditional) distinguishing number were given independently by Klavžar, Wong, and Zhu [36] and Collins and Trenk [19]. We give the result as stated by Klavžar, Wong, and Zhu.

Theorem 4.2 ([36]) *Let G be a connected graph. Then $D(G) \leq \Delta(G)$ unless G is either K_n, $n \geq 1$, $K_{n,n}$, $n \geq 1$, or C_5. In these cases $D(G) = \Delta(G) + 1$.*

The proof of Theorem 4.2 uses the following lemma, which we will use as well.

Lemma 4.3 ([36]) *Suppose (G, ℓ) is a connected, vertex colored graph such that $\ell(v)$ is the color for $v \in V(G)$. Let every vertex of the set $X \subseteq V(G)$ be fixed by every automorphism of (G, ℓ). Let $x \in X$ and set $S = N_G(x) \setminus X$. If $\ell(u) \neq \ell(v)$ for any pair of distinct vertices u and v in S, then every vertex of S is fixed by every automorphism of (G, ℓ).*

We are able to modify the approach from [36] to give a Brooks-type theorem for list-distinguishing colorings. Interestingly, we are able to show that the traditional distinguishing problem is precisely what prevents the exceptional graphs K_n, $K_{n,n}$ and C_5 from being Δ-list-distinguishable.

Proposition 4.4 *Let G be a connected graph and let $\mathcal{L} = \{L(v)\}$ be an assignment of lists of size $\Delta(G)$ to $V(G)$. Then G can be $\mathcal{L}$-distinguished unless G is one of K_n, $K_{n,n}$, or C_5 and $|\bigcup L(v)| = \Delta(G)$. In these exceptional cases, G can be colored from any assignment of lists of length $\Delta(G) + 1$.*

Proof: Assign list $L(v)$ to each vertex $v \in V(G)$ such that $|L(v)| = \Delta(G)$, and assume $|\bigcup L(v)| \neq \Delta(G)$. Since G is connected, there exist two vertices, x and y, such that $L(x) \neq L(y)$ and $xy \in E(G)$. Let $c_x \in L(x) - L(y)$, and color x with c_x. Going forward, no other vertex but x will receive color c_x, assuring that x will be fixed by every color-preserving automorphism of G. Construct a breadth first search spanning tree of G rooted at x. Since $|N_G(x)| \leq \Delta(G)$, we can color each vertex $w \in N_G(x) - y$ with a unique color, $c_w \in L(w)$, such that $c_w \neq c_x$. Since $|L(y)| = \Delta(G)$ and $c_x \notin L(y)$, there exists a color, $c_y \in L(y)$ such that $c_y \neq c_w$ for all $w \in N_G(x) - y$; color y with c_y. From here each vertex has at most $\Delta(G) - 1$ children in the spanning tree. Therefore we can color each

sibling of a vertex in the spanning tree uniquely from its list, never using the color c_x.

Since x is the only vertex colored c_x, all color-preserving automorphisms must map x to itself. By Lemma 4.3 and that every pair of vertices in the neighborhood of x has a distinct color, each vertex in the neighborhood of x must also map to itself in a color-preserving automorphism. Similarly, we apply Lemma 4.3 to each level of the breadth first search tree, forcing each vertex to be mapped to itself in a color-preserving automorphism. Thus the coloring so constructed is a list-distinguishing coloring of G.

It remains to consider when $|\bigcup L(v)| = \Delta(G)$, which is the same as distinguishing coloring. By [19] and [36], if G is K_n, $K_{n,n}$, or C_5 then $D(G) = \Delta(G)+1$ and if not then $D(G) = \Delta(G)$. Since in this case the lists are all the same, we can color the vertices as in the distinguishing colorings. Thus if G is K_n, $K_{n,n}$, or C_5 and $|\bigcup L(v)| = \Delta(G)$, then $D_\ell(G) = \Delta(G) + 1$; and if G is not K_n, $K_{n,n}$, or C_5 and $|\bigcup L(v)| = \Delta(G)$, then $D_\ell(G) = \Delta(G)$. ∎
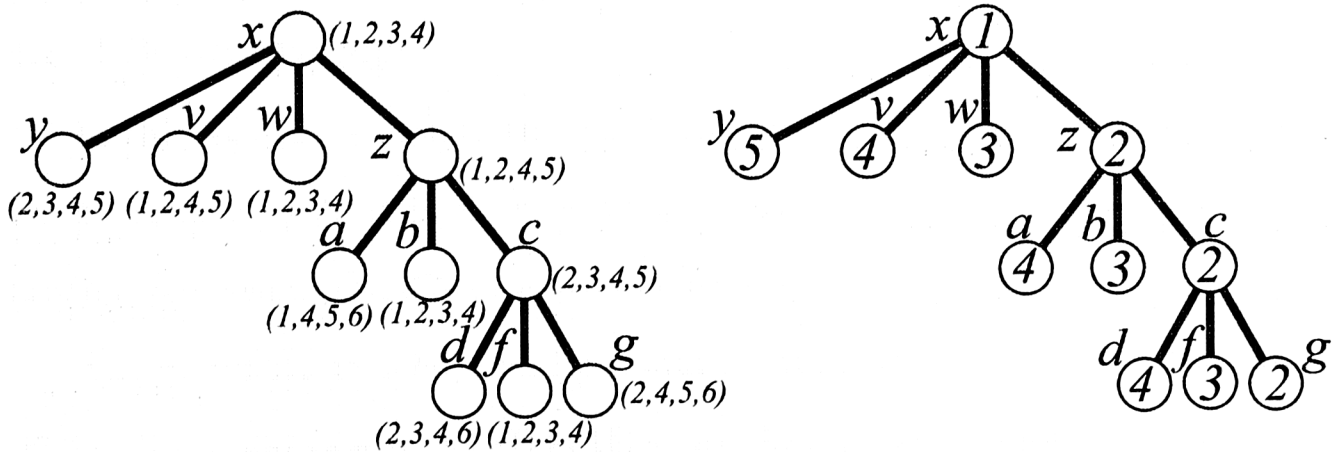
Figure 4.1: A labeled picture to illustrate Proposition 4.4.

Figure 4.1 gives a breadth first search spanning tree to illustrate the proof of Proposition 4.4. Each vertex is assigned a list of length 4, which is the maximum degree. The graph on the left gives an example of one such assignment of lists. Vertex x is chosen as the root of the tree, because it has an adjacency, y, that does not have the same list as x. Since 1 is not in the list for y, we can color all of the children of x differently, without using 1. The descendants of x will all have at most three children in the tree, ensuring each child can be colored uniquely without using 1 with a list of length 4. Notice that the vertex g is colored the same as two of its ancestors; however, as long as it is colored differently than d and f, Lemma 4.3 ensures that the coloring fixes those vertices under all automorphisms.

Theorem 4.4 immediately yields the following Brooks-type result.

Theorem 4.5 *Let G is a graph with maximum degree Δ. Then $D_\ell(G) \leq \Delta$ unless G is one of C_5, K_n or $K_{n,n}$, in which case $D_\ell(G) = \Delta + 1$.*

4.2 Dihedral Groups

In this section, we study the list-distinguishing number of graphs realizing D_n for some $n \geq 3$. It is clear that C_n realizes D_n, but there are many other graphs that realize the dihedral group (see Figure 4.2).

The group D_n is generated by σ and τ, where $\sigma^n = e$, $\tau^2 = e$, and $\tau\sigma = \sigma^{n-1}\tau$. It is important to note that there are three types of non-trivial subgroups of D_n. The first is $\langle\sigma^j\rangle$, which is a cyclic subgroup. The second is $D_m = \langle\sigma^j, \tau\sigma^i\rangle$, where m divides n, and the third is generated by an element of order two, $\langle\tau\sigma^i\rangle$.
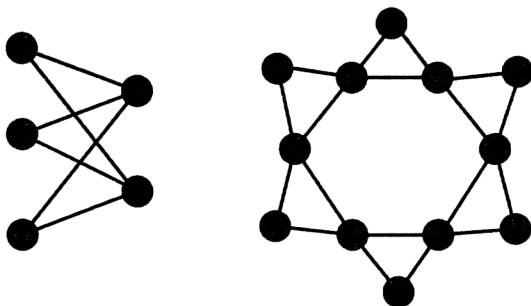
Figure 4.2: Two graphs that realize D_6.

As previously mentioned, Albertson and Collins introduced this topic by examining the distinguishing number of graphs that realize the dihedral group.

Theorem 4.6 *[2] If G realizes D_n then $D(G) = 2$ unless $n = \{3, 4, 5, 6, 10\}$ in which case $D(G)$ is either 2 or 3.*

The main result of this section is as follows.

Theorem 4.7 *Let G be a graph realizing D_n. Then $D(G) = D_\ell(G)$.*

We point out at this time that the lemmata developed here to prove Theorem 4.7 appear nearly identical to those utilized to prove the corresponding theorem in [2]. However, the techniques used here frequently vary greatly from those in [2], illustrating further the distinctions between the distinguishing and list-distinguishing numbers.

Lemma 4.8 *Let Γ be a group and G be a graph realizing Γ, and suppose $u_1, \ldots, u_t$ are vertices from different vertex orbits of G. If $\bigcap_{i=1}^{t} St(u_i) = \{e\}$, then $D_\ell(G) = 2$.*

Proof: For each u_i, select a color $c_i \in L(u_i)$ and then color each other vertex in $O(u_i)$ with any color other than c_i. Let g be any nonidentity element in Γ. Since $\bigcap_{i=1}^{t} St(u_i) = \{e\}$, at least one u_i is not fixed by G. Since $u_i g$ is not colored c_i, this is a 2-list-distinguishing coloring. ∎
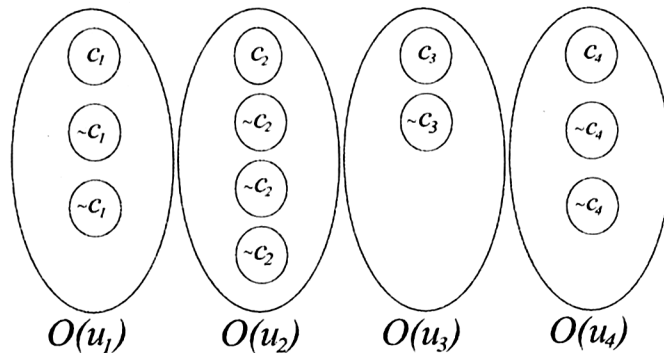
Figure 4.3: An illustration of Lemma 4.8.

Figure 4.3 gives an illustration of the proof of Lemma 4.8, where $\sim c_x$ means any color except c_x. The general idea of this lemma is that the only automorphism that simultaneously fixes each of $u_1, \ldots, u_t$ to themselves is the trivial automorphism, and vertices are only mapped to other vertices in their orbit. Thus coloring $u_1, \ldots, u_t$ uniquely in their respective orbits, which can easily be done with lists of length 2, will force any color-preserving automorphism to fix these vertices, and the only automorphism that fixes all of these vertices is the trivial automorphism.

The following lemmas, the first of which appears in [2], will be useful as we proceed.

Lemma 4.9 *Let G realize D_n, and suppose that G has t vertex orbits. If*

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$u_1, \ldots, u_t$ *are vertices from the t different vertex orbits of G, then*

$$\langle \sigma \rangle \cap St(u_1) \cap \cdots \cap St(u_t) = \{e\}.$$

Proof: The proof of this lemma is identical to the proof of the corresponding lemma in [2]. Since $\sigma^l \sigma^t \sigma^{-l} = \sigma^t$ and $\tau \sigma^i \sigma^t \tau \sigma^i = \sigma^{n-t}$, the conjugacy class of σ^t in D_n is $\{\sigma^t, \sigma^{n-t}\}$. Thus if σ^t is an element of any subgroup H, then σ^t is an element of any subgroup conjugate to H. Hence if $\sigma^t \in St(u_1) \cap \cdots \cap St(u_t)$, then σ^t is in every conjugate of each of these stabilizers, which implies σ^t is in every stabilizer of every vertex of G. If σ^t fixes every vertex of G, then $\sigma^t = e$ and $t = n$, since G realizes D_n.

∎

Lemma 4.10 *Let G realize D_n. If there is a vertex u in G such that $St(u) = \langle \sigma^j \rangle$ then $D_\ell(G) = 2$.*

Proof: The proof of this lemma is identical to the proof of the corresponding lemma in [2]. Let $u_1, \ldots, u_t$ be vertices from each of the different orbits of G. Then $\bigcap_i St(u_i) \subseteq St(u) = \langle \sigma^j \rangle$, which implies that $\bigcap_i St(u_i) = \{e\}$ by Lemma 4.9, thus completing the proof by Lemma 4.8. ∎

At this point we begin to more seriously modify the techniques from [2] in order to better fit our list-coloring framework.

The intuition for the following proof comes from Proposition 1.6. Although an orbit of the form $O(u) = \{u, u\sigma, u\sigma^2, \ldots, u\sigma^{j-1}\}$ need not be a cycle of length j, the movement of this orbit can be envisioned as such, which gives us a working mental picture for the proof (see Figures 4.4 and 4.5) as we verify all the details.

Lemma 4.11 *Let G realize D_n and let u be a vertex in G such that $St(u)$ is either $\langle \sigma^j, \tau\sigma^i \rangle$ or $\langle \tau\sigma^i \rangle$. If $|O(u)| \geq 6$, then $O(u)$ is 2-list-distinguishable.*

Proof: If vertex u has stabilizer $\langle \sigma^j, \tau\sigma^i \rangle$, then $O(u) = \{u, u\sigma, u\sigma^2, \ldots, u\sigma^{j-1}\}$. If u has stabilizer $\langle \tau\sigma^i \rangle$, then $O(u) = \{u, u\sigma, u\sigma^2, \ldots, u\sigma^{n-1}\}$. Either way, we have that $O(u) = \{u, u\sigma, u\sigma^2, \ldots, u\sigma^{j-1}\}$ for some j, and hence we assume that $j \geq 6$. Consider the set $A = \{u, u\sigma^2, u\sigma^3\}$ and select a color $c_u \in L(u)$. If possible, color $u\sigma^2$ and $u\sigma^3$ with c_u as well. We will demonstrate that it is possible to extend this coloring to a 2-list-distinguishing coloring without using c_u on any vertex in $V(G) - A$.

Case 1: All three vertices in A are colored with c_u.

We proceed by coloring the vertices in $V(G) - A$ using any color in their respective lists except c_u. Any automorphism g in D_n that fixes this coloring of $O(u)$ must permute the vertices in A, and specifically must map u to some element of A. If $ug = u$, then $g \in St(u)$; if $ug = u\sigma^2$, then $g \in St(u)\sigma^2$; and if $ug = u\sigma^3$, then $g \in St(u)\sigma^3$. As was demonstrated in [2], the only automorphisms from these sets that permute A actually fix all of $O(u)$. We will include this argument for completeness.

Suppose $g \in D_n$ fixes A, so $ug = u, u\sigma^2$, or $u\sigma^3$. The automorphisms that send u to u are $St(u)$, the automorphisms that send u to $u\sigma^2$ are $St(u)\sigma^2$, and the automorphisms that send u to $u\sigma^3$ are $St(u)\sigma^3$. Now we show that each automorphism in $St(u), St(u)\sigma^2, St(u)\sigma^3$ either acts trivially on $O(u)$ or does not fix A. The subgroup of automorphisms that act trivially on $O(u)$ is the intersection of the stabilizer subgroups of each vertex, so if $St(u) = \langle \sigma^j, \tau\sigma^i \rangle$ this is $\langle \sigma^j \rangle$ and if $St(u) = \langle \tau\sigma^i \rangle$ this is e. If $St(u) = \langle \sigma^j, \tau\sigma^i \rangle$, then j divides n

by the Orbit/Stabilizer Theorem, and the elements of $St(u)$ are

$$e, \sigma^j, \sigma^{2j}, \ldots, \sigma^{(\frac{n}{j}-1)j}, \tau\sigma^i, \tau\sigma^{i+j}, \ldots, \tau\sigma^{i+(\frac{n}{j}-1)j}.$$

If $0 \le d \le \frac{n}{j} - 1$, then below are the outcomes when the automorphisms above are applied to A.

$$A\sigma^{dj} = A \qquad\qquad A\tau\sigma^{i+dj} = \{u, u\sigma^{n-2}, u\sigma^{n-3}\}$$

$$A\sigma^{dj}\sigma^2 = \{u\sigma^2, u\sigma^4, u\sigma^5\} \quad A\tau\sigma^{i+dj}\sigma^2 = \{u\sigma^2, u, u\sigma^{n-1}\}$$

$$A\sigma^{dj}\sigma^3 = \{u\sigma^3, u\sigma^5, u\sigma^6\} \quad A\tau\sigma^{i+dj}\sigma^3 = \{u\sigma^3, u\sigma, u\}$$

Since $n \ge 6$, the only automorphism that maps A to A is σ^{dj}, which acts trivially on $O(u)$. If $St(u) = \langle \tau\sigma^i \rangle$, then $St(u) = \{e, \tau\sigma^i\}$, and below are the outcomes when these automorphisms are applied to A.

$$A\tau\sigma^i = \{u, u\sigma^{n-2}, u\sigma^{n-3}\}$$

$$A\tau\sigma^i\sigma^2 = \{u\sigma^2, u, u\sigma^{n-1}\}$$

$$A\tau\sigma^i\sigma^3 = \{u\sigma^3, u\sigma, u\}$$

None of these automorphisms map A to A. Thus this coloring is a 2-list-distinguishing coloring of $O(u)$.

Case 2: The vertex u is the only one in A colored with c_u.

Suppose that $St(u) = \langle \sigma^j, \tau\sigma^i \rangle$. Then j divides n by the Orbit/Stabilizer Theorem, so the assumption that $j \ge 6$ implies that $u\sigma^2 \ne u\sigma^{n-2}$, which also holds when $St(u) = \langle \tau\sigma^i \rangle$. In either case, we extend the coloring of A by first assigning the vertex $u\sigma^{n-2}$ any color $c \ne c_u$ in $L(u\sigma^{n-2})$. Since we utilize the color c_u on the vertices in A wherever possible, and u is the only vertex of G that receives color c_u, we conclude that $c_u \notin L(u\sigma^2)$. Therefore there is a color

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c' in $L(u\sigma^2)$, different from both c and c_u. Color $u\sigma^2$ using color c' and color the remaining vertices of G with any color aside from c_u in their respective lists.

We now show that this coloring distinguishes $O(u)$. Since u is the unique vertex of color c_u, any color-preserving automorphism $g \in D_n$ must lie in $St(u)$. Then either $g = \sigma^{dj}$ or $g = \tau\sigma^{i+dj}$, with d being necessarily zero when $St(u) = \langle \tau\sigma^i \rangle$. Note that σ^{dj} fixes $O(u)$ and that, for any d, $\tau\sigma^{i+dj}$ takes $u\sigma^2$ to $u\sigma^{n-2}$. Since we have constructed our coloring so that $u\sigma^2$ and $u\sigma^{n-2}$ have different colors, this yields a 2-list-distinguishing coloring of $O(u)$.
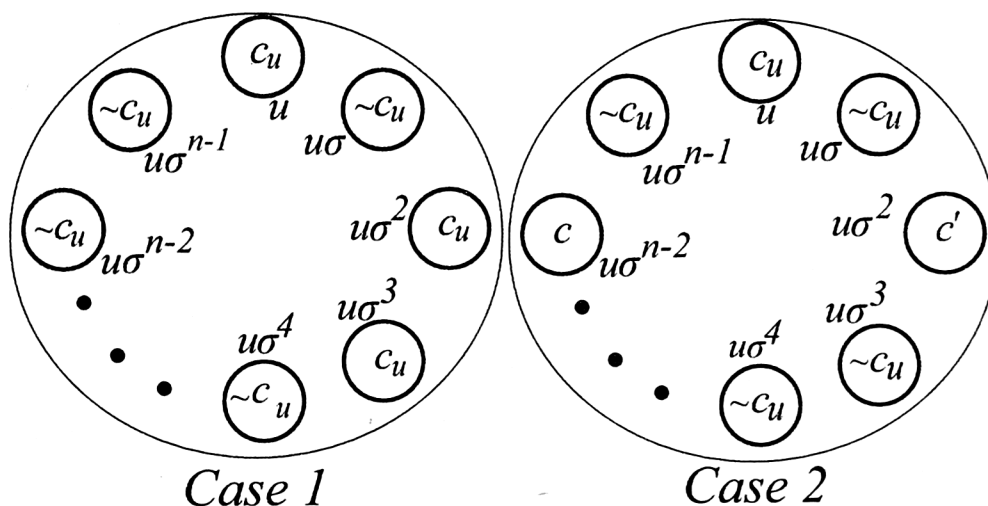
Figure 4.4: An illustration of Case 1 and Case 2 of Lemma 4.11, where $c \neq c'$.

Case 3: The vertices u and $u\sigma^2$ are the only ones in A colored with c_u.

As above, if $St(u) = \langle \sigma^j, \tau\sigma^i \rangle$, then j divides n by the Orbit/Stabilizer Theorem. Consequently, the assumption that $j \geq 6$ implies that $u\sigma^3 \neq u\sigma^{n-1}$, which again also holds when $St(u) = \langle \tau\sigma^i \rangle$. In this case, we extend the coloring of u and $u\sigma^2$ in a similar manner to Case 1, with two exceptions. We make no special color assignment to $u\sigma^{n-2}$, save the standard assumption that it does not receive color c_u. Instead, we assign different colors to the vertices $u\sigma^3$ and

$u\sigma^{n-1}$ such that neither vertex is colored with c_u. As above, this is possible since $c_u \notin L(u\sigma^3)$, or else we would have used it to color $u\sigma^3$.

Now every automorphism $g \in D_n$ that preserves the colors on $O(u)$ either interchanges u and $u\sigma^2$ or fixes both of these vertices. As discussed in the previous case, any element of $St(u)$ will either fix all of $O(u)$ or will map $u\sigma^2$ to $u\sigma^{n-2}$. Thus as $u\sigma^{n-2}$ is not colored with c_u, any color preserving automorphism g must interchange u and $u\sigma^2$, which implies that g has the form $g'\sigma^2$, where g' is an element of $St(u)$. Specifically, either $g' = \sigma^{dj}$ or $g' = \tau\sigma^{i+dj}$ for some integer d, with $d = 0$ if $St(u) = \langle \tau\sigma^i \rangle$. In either case, $g'\sigma^2$ takes $u\sigma^3$ to $u\sigma^{n-1}$, implying that this coloring 2-list-distinguishes $O(u)$.

Case 4: The vertices u and $u\sigma^3$ are the only ones in A colored with c_u.

Suppose first that $St(u) = \langle \tau\sigma^i \rangle$. We wish, as above, to extend our coloring of A to a list-distinguishing coloring of $O(u)$ in which no vertices aside from u and $u\sigma^3$ receive color c_u. Any element $g \in D_n$ that fixes such a coloring must either stabilize u or exchange u and $u\sigma^3$. Consequently, either $g = \tau\sigma^i$ or $g = \tau\sigma^{i+3}$, so consider the outcomes when these elements are applied to A:

$$A\tau\sigma^i = \{u, u\sigma^{n-2}, u\sigma^{n-3}\}$$

$$A\tau\sigma^{i+3} = \{u\sigma^3, u\sigma, u\}.$$

Therefore, we would like to choose colors $x \in L(u\sigma)$ and $y \in L(u\sigma^{n-2})$ such that $x, y \neq c_u$ and choose a color $z \neq x, y$ from $L(u\sigma^2)$. We would then extend our coloring of A by assigning these colors to their respective vertices and then coloring the remaining vertices in G using any color except c_u. The coloring so described is possible unless all of the following hold: $L(u\sigma) = \{c_u, x\}$, $L(u\sigma^{n-2}) = \{c_u, y\}$ and $L(u\sigma^2) = \{x, y\}$. In this case, suppose $L(u) = \{c_u, c_u'\}$. We will then recolor

u with c'_u and again color $u\sigma^2$ and $u\sigma^3$ with these colors if possible. Then the analysis conducted thus far assures that we can construct a 2-list-distinguishing coloring of $O(u)$, as we cannot have $L(u\sigma) = \{c'_u, x\}$, $L(u\sigma^{n-2}) = \{c'_u, y\}$ and $L(u\sigma^2) = \{x, y\}$ for $x = y$, and this was the only obstacle preventing us from constructing the desired coloring when $St(u) = \langle \tau\sigma^i \rangle$.

Hence we assume that $St(u) = \langle \sigma^j, \tau\sigma^i \rangle$. In this case, j divides n by the Orbit/Stabilizer Theorem, and we may assume that $n > j \geq 6$. Again, we wish to extend our coloring of A to a distinguishing coloring of G in which no vertices aside from u and $u\sigma^3$ receive color c_u. If $g \in D_n$ fixes such a coloring then either $g \in St(u)$ or $g = g'\sigma^3$ for some $g' \in St(u)$. Thus for some $d \geq 0$, $g = \sigma^{dj}$, either $\sigma^{dj+3}, \tau\sigma^{i+dj}$ or $\tau\sigma^{i+dj+3}$. Applying each of these elements to A, we obtain:

$$A\sigma^{dj} = A$$
$$A\sigma^{dj+3} = \{u\sigma^3, u\sigma^5, u\sigma^6\}$$
$$A\tau\sigma^{dj+i} = \{u, u\sigma^{n-2}, u\sigma^{n-3}\}$$
$$A\tau\sigma^{dj+i+3} = \{u\sigma^3, u\sigma, u\}.$$

Since j divides n and n is at least seven, $u\sigma^{n-3} \neq u\sigma^3$. Therefore $g = \tau\sigma^{dj+i}$ cannot preserve our proposed coloring. Additionally, σ^{dj} fixes all of $O(u)$, so we need only consider the cases where $g = \sigma^{dj+3}$ or $g = \tau\sigma^{dj+i+3}$.

If $j > 6$, then since σ^{dj+3} takes $u\sigma^3$ to $u\sigma^6$ this choice of g cannot distinguish u. Thus when $j > 6$ is suffices to assume that $g = \tau\sigma^{i+dj}$ and thus we need only to distinguish $u\sigma$ from $u\sigma^2$ without assigning c_u to $u\sigma$, which is clearly possible as we have assumed that $c_u \notin L(u\sigma^2)$.

Thus we assume that $j = 6$ and that either $g = \sigma^{dj}$ or $g = \tau\sigma^{i+dj+3}$. We would like to choose colors $x \in L(u\sigma)$ and $y \in L(u\sigma^5)$ such that $x, y \neq c_u$ and

choose a color $z \neq x, y$ from $L(u\sigma^2)$. Then we would extend our coloring of A by assigning these colors to their respective vertices and then coloring the remaining vertices in G using any color except c_u. As above, this coloring is possible unless $L(u\sigma) = \{c_u, x\}$, $L(u\sigma^5) = \{c_u, y\}$ and $L(u\sigma^2) = \{x, y\}$. However, in this case we proceed by changing our initial coloring of A by using the color $c'_u \neq c_u$ in $L(u)$. As when $St(u) = \langle \tau\sigma^i \rangle$, this allows us to construct a coloring that will distinguish $O(u)$, and therefore completes the proof. ∎

Case 3   Case 4

Figure 4.5: An illustration of Cases 3 and 4 of Lemma 4.11, where $c \neq c'$ and $x, y \neq z$.

Now that we have shown that sufficiently large orbits can be 2-list-distinguished, we show that if a graph has one of these large orbits that the 2-list-distinguishing coloring on the orbit can be extended to 2-list-distinguish the entire graph.

The following is a slight modification of the corresponding lemma in [2].

Lemma 4.12 *Let G realize D_n. Let u be a vertex such that $u \in V(G)$ and $St(u) = \langle \sigma^j, \tau\sigma^i \rangle$ or $\langle \tau\sigma^i \rangle$. If $|O(u)| \geq 6$, then $D_\ell(G) = 2$.*

Proof: Assign lists of length two to each vertex in G. If $St(u) = \langle \tau\sigma^i \rangle$, then the intersection of the subgroups conjugate to $St(u)$ is the identity. Applying

Lemma 4.11, $O(u)$ is 2-list-distinguishable and thus by Lemma 4.8, $D_\ell(G) = 2$.

Therefore, assume $St(u) = \langle \sigma^j, \tau\sigma^i \rangle$. Since $O(u)$ is 2-list-distinguishable, we need only consider the automorphisms that act trivially on $O(u)$. These are the intersection of the stabilizers of vertices of $O(u)$, which is $\Lambda = \langle \sigma^j \rangle$. The group action of Λ on G creates vertex orbits $U_1, U_2, .., U_t$. The orbit $O(u)$ is broken into 1-orbits under the group action Λ, because σ^j fixes $O(u)$. From each orbit U_i such that $|U_i| > 1$, select a vertex u_i and color it with any color $c_i \in L(u_i)$; notice we are not recoloring the vertices of $O(u)$. Then color the remaining vertices of U_i with any color other than c_i, construct a list-distinguishing coloring of $O(u)$ from its assigned lists, and color all other uncolored vertices in G with any color from their respective lists. If a nontrivial automorphism in Λ fixes u_i, then it must fix all of U_i. Thus each $g \neq e$ in Λ map move some u_i to another vertex in its orbit, implying that the only color preserving automorphism is the identity. Consequently, this is a 2-list-distinguishing coloring of G, so $D_\ell(G) = 2$. ∎

Lemmas 4.8 - 4.11 provide the necessary machinery to complete the proof of Theorem 4.7, and we do so now.

Proof: (of Theorem 4.7) Albertson and Collins proved in [2], that if n has a divisor which is a prime power p^α greater than 6 there must be an orbit of order at least 6. By Lemma 4.2 these graphs are 2-list-distinguishable, and so the only remaining cases are $n = 3, 4, 5, 6, 10, 12, 15, 20, 30, 60$. Assume that $n \geq 12$ and that every orbit has size less than or equal to 5. By Lemma 4.10, we may assume that the stabilizer of every vertex is either $\langle \sigma^j, \tau\sigma^i \rangle$ of order $\frac{2n}{j}$ or $\tau\sigma^i$ of order 2. By the Orbit/Stabilizer Theorem, an stabilizer of order 2 corresponds to an orbit of size n, which is greater than 5. Hence each vertex

must have the stabilizer $\langle \sigma^j, \tau \sigma^i \rangle$.

For G to realize D_n, G must have orbits with sizes whose least common multiple is n. Since we are assuming that all orbits are smaller than 6, if $n = 12$, there are orbits of size $3, 4$; if $n = 15$, there are orbits of size $3, 5$; if $n = 20$, there are orbits of size $4, 5$; if $n = 30$, there are orbits of size $2, 3, 5$; and if $n = 60$, there are orbits of size $3, 4, 5$. Since 15 is odd, there is no 2-orbits if $n = 15$; since 4 doesn't divide 30, there is no 4-orbit if $n = 30$. If $n = 12, 20$, or 60, then there is no 2-orbit, or we could choose a stabilizer from the 2-orbit and 4-orbit whose intersection is $\langle \sigma^4 \rangle$. Then by Lemma 4.9, the intersection of the stabilizers from all orbits is the identity, and by Lemma 4.8 we can 2-list distinguish G. Thus we can assume that, except for the 1-orbits, the orbits for each n must be exactly as listed in the second sentence of this paragraph.

Therefore if $n \geq 12$, the orbits of G have sizes which are all pairwise relatively prime. From [2], D_n is the product of the automorphism group of each orbit, considered as a subgraph of G. Since each orbit must be a vertex transitive graph, the orbits of size 3 are K_3 or its complement; the orbits of size 4 must be either K_4 or its complement or C_4 or its complement; and the orbits of size 5 must be either K_5 or its complement or C_5 or its complement. In each case, the product of the sizes of the appropriate groups is larger than the size of D_n for each value of n. Thus if $n \geq 12$ it must have an orbit of size 6 or larger, and G can be 2-list-distinguished by Lemma 4.2.

It remains to show that the theorem holds when $n = 3, 4, 5, 6$ or 10. Suppose first that G realizes D_n, where $n = 3$ or 5 and select a vertex u with nontrivial orbit in G. By Lemma 4.10, we may assume that every vertex u in G has

$St(u) = \langle \tau\sigma^i \rangle$ or $\langle \sigma^j, \tau\sigma^i \rangle$. However, as n is prime, if σ^j fixes u, then $j = 0$ or $j = 1$. Consequently, we may assume that $j = 0$, as $\langle \sigma, \tau\sigma^i \rangle = D_n$, and by assumption u has a nontrivial orbit.

Note as well that if $\tau\sigma^i$ is in $St(u)$, then τ is in $St(u\sigma^x)$ where $2x \equiv n - i$ (mod n). Such an x exists for all i when $n = 3$ or 5, so we may assume that $St(u) = \langle \tau \rangle$. Then, since the orbit of u is by assumption nontrivial, $O(u) = \{u, u\sigma, \ldots, u\sigma^{n-1}\}$ and the only element of D_n that fixes all of $O(u)$ is e.

If G has exactly one nontrivial orbit, then this orbit is precisely $O(u) = \{u, u\sigma, \ldots, u\sigma^{n-1}\}$. Since this orbit behaves as C_3 or C_5, we need 3 distinct colors to distinguish $O(u)$ and thus G; therefore, $D(G) = D_\ell(G) = 3$ (see G_1 in Figure 4.6). Suppose then that G has vertices u and v with distinct nontrivial orbits and that we have also assigned lists of length 2 to the vertices of G. We may furthermore assume, via the above discussion, that $St(u) = St(v) = \langle \tau \rangle$. If it is possible to 2-list-distinguish $O(u)$, then we do so and color the rest of the graph as in the proof of Lemma 4.2. If it is not possible to color each vertex in $O(u)$ with a unique color, then each vertex in $O(v)$ must have list $L(u\sigma^i) = \{c_1, c_2\}$. We then color u with color c_1 and color both $u\sigma$ and $u\sigma^2$ with color c_2. If g, a nonidentity element of D_n, fixes this coloring of $O(u)$, then $g \in St(u)$. However, since $O(v)$ is nontrivial, g must exchange two elements in $O(v)$, say $v\sigma^i$ and $v\sigma^j$. Assigning these vertices distinct colors from their respective lists serves to 2-list-distinguish G (see G_2 in Figure 4.6), and thus implies that $D(G) = D_\ell(G) = 2$.

Next let G be a graph that realizes D_4 and furthermore assume that there is no vertex u in G such that $St(u) = \langle \sigma^j \rangle$ or $|O(u)| \geq 6$. Lemmas 4.8 and

Figure 4.6: Two graphs that realize D_5 with example lists and list-distinguishing colorings using white, black, and grey.

4.9 imply that if there are vertices u and v in distinct orbits of G such that $\tau\sigma^i \in St(u)$ but $\tau\sigma^i \notin St(v)$, then $D(G) = D_\ell(G) = 2$ since the intersection of the stabilizers of u and v and $\langle\sigma\rangle$ will be trivial.

Suppose that τ stabilizes some element in every orbit of G (the case where some other $\tau\sigma^i$ stabilizes an element in every orbit is handled similarly). Under our assumptions, every nontrivial orbit in G must have order either two or four. If every nontrivial orbit has two elements, then τ stabilizes every vertex in G, a contradiction. Therefore, there is some vertex u such that $St(u) = \langle\tau\rangle$ and $O(u) = \{u, u\sigma, u\sigma^2, u\sigma^3\}$. If possible, color the orbit of u in a manner consistent with the traditional distinguishing coloring of C_4. Specifically, for some i, color $u\sigma^i$ and $u\sigma^{i+1}$ with the same color c and then color the other two vertices in $O(u)$ with distinct colors other than c. Since no element of D_4 fixes all of $O(u)$, this would suffice to 2-list-distinguish G and also shows that $D_\ell(G) \leq 3$. Thus we may assume that $L(u\sigma^i) = \{c_1, c_2\}$ for all i implying that if $O(u)$ is the only nontrivial orbit of G, $D(G) = D_\ell(G) = 3$.

Next assume that there is some vertex v, not in $O(u)$, such that $O(v)$ is also nontrivial and assign lists of length two to each vertex in G. We claim that it is possible to construct a list-distinguishing coloring of G from these

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lists. If $|O(v)| = 4$, then $O(v) = \{v, v\sigma, v\sigma^2, v\sigma^3\}$ and we may assume that $L(v\sigma^i) = \{c'_1, c'_2\}$ for all i. Without loss of generality, suppose that τ stabilizes v (and therefore $v\sigma^2$). Color u and $u\sigma$ with color c_1, $u\sigma^2$ and $u\sigma^3$ with color c_2, v and $v\sigma^2$ with color c'_1 and, finally, $v\sigma$ and $v\sigma^3$ with color c'_2. Then the only automorphism that fixes the colors in both $O(u)$ and $O(v)$ is e, so this coloring 2-list-distinguishes G (see G_1 in Figure 4.7).

We may therefore suppose that $O(v)$, and every nontrivial orbit of G aside from $O(u)$, has exactly two elements, and hence that $St(v) = \langle \sigma^2, \tau \rangle$ and $O(v) = \{v, v\sigma\}$. The case where $St(v) = \langle \sigma^2, \tau\sigma^j \rangle$ is again similar. We will color u and $u\sigma$ with color c_1, $u\sigma^2$ and $u\sigma^3$ with color c_2 and color v and $v\sigma$ with distinct colors. The only nonidentity automorphism of G that preserves this coloring of $O(u)$ is $\tau\sigma^3$. However, this interchanges the elements of $O(v)$, which received different colors, so this coloring 2-list-distinguishes G (see G_2 in Figure 4.7).
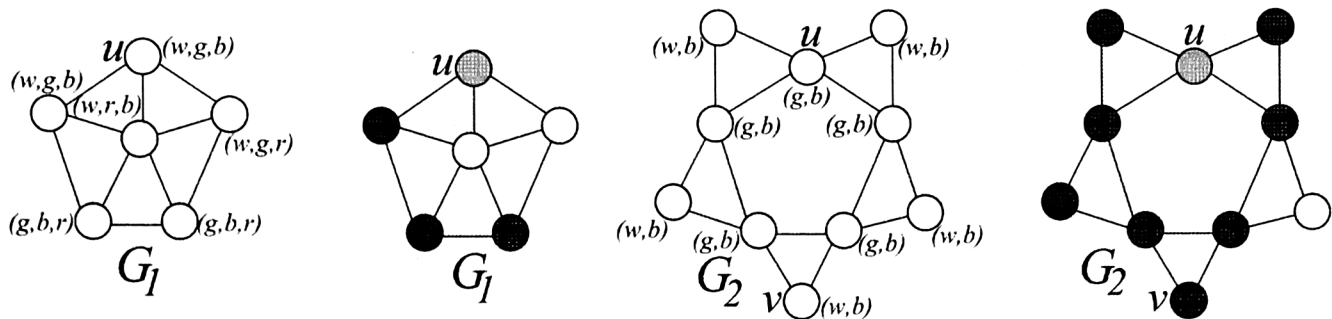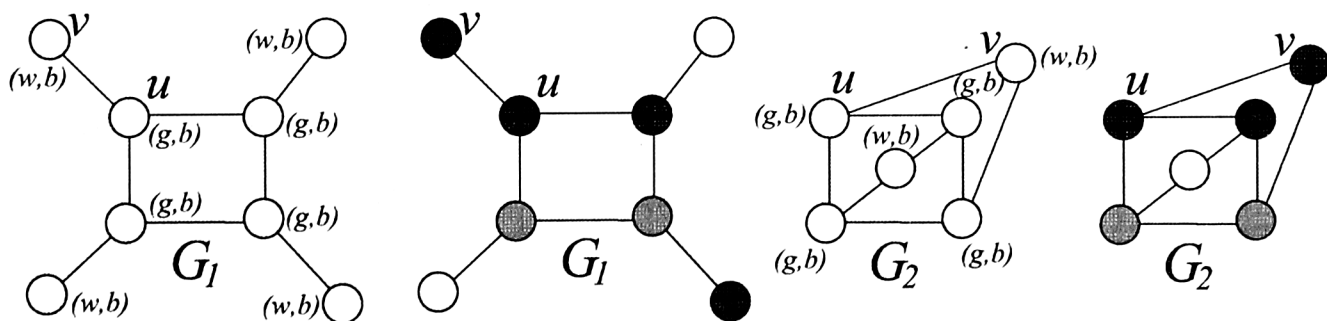

Figure 4.7: Two graphs that realize D_4 with example lists and list-distinguishing colorings using white, black, and grey.

Next we consider the penultimate case, that G realizes D_6. By Lemmas 4.10 and 4.2, we may assume that there is no vertex u in G has $St(u) = \langle \sigma^j \rangle$ or $|O(u)| \geq 6$. If there exists a $u \in V(G)$ such that $St(u) = \langle \tau\sigma^i \rangle$, then $|O(u)| = 6$, therefore we assume that every vertex u in G has $St(u) = \langle \sigma^j, \tau\sigma^i \rangle$. Given this stabilizer and the Orbit/Stabilizer Theorem, for every vertex u in

G, $|O(u)| = 1, 2$ or 3. This implies that G can easily be 3-list-distinguished, as it is possible to color all of the vertices in a given orbit with distinct colors. Furthermore, if there is no orbit of order 3, then it is not difficult to 2-list-distinguish G.

Therefore, let us first assume there is exactly one orbit of order 3. Specifically, let u be a vertex such that $|O(u)| = 3$ and observe that necessarily $O(u) = \{u, u\sigma, u\sigma^2\}$. Suppose first that for each $x \in O(u)$ there is some $\phi_x \in Aut(G) \cap St(x)$ such that the following hold:

1. ϕ_x interchanges the two vertices in $O(u) - x$, and

2. ϕ_x fixes all of $V(G) - O(u)$.

In this case, we cannot 2-distinguish G with any 2-coloring, as without loss of generality both $u\sigma$ and $u\sigma^2$ will receive the same color. However, then the automorphism ϕ_u described above is nontrivial and color-preserving, regardless of how the remainder of G is colored.

If there is some $x \in O(u)$ for which no such ϕ_x exists, then we claim that G is 2-list-distinguishable (and hence 2-distinguishable). In this case, we assign distinct colors to all pairs of vertices lying in orbits of order two and also color all vertices of $O(u)$ with distinct colors, if this is possible. If it is not possible to color $O(u)$ in this way, then each vertex in $O(u)$ must be assigned identical lists, say $\{c_1, c_2\}$. We then color x with c_1 and the vertices in $O(u) - x$ with c_2. Due to the assumptions that $O(u)$ is the unique orbit of order three and that there is no ϕ_x as described above, any nontrivial automorphism $g \in Aut(G)$ that preserves this coloring of $O(u)$ must interchange the elements of an orbit

of order two. However, all such orbits have been colored with distinct colors, so g is not color-preserving. It follows that $D_\ell(G) = 2$

Suppose then that G has more than one orbit of order 3, and assign lists of order 2 to the vertices of G. Choose vertices u and v in distinct orbits such that $|O(u)| = |O(v)| = 3$. If it is not possible to color each vertex in $O(u)$ with a distinct color (as in G_1 in Figure 4.8), then each vertex in $O(u)$ must have the same list, $\{c_1, c_2\}$. We then color u with c_1 and the other two vertices with color c_2. If g, a nonidentity element of D_6, fixes this coloring of $O(u)$, then $g \in St(u)$. However, since $|O(v)| = 3$, g must exchange two elements in $O(v)$, call them v' and v''. Assigning v' and v'' distinct colors from their respective lists serves to 2-list-distinguish G (see G_2 in Figure 4.8). Therefore, $D(G) = D_\ell(G) = 2$.
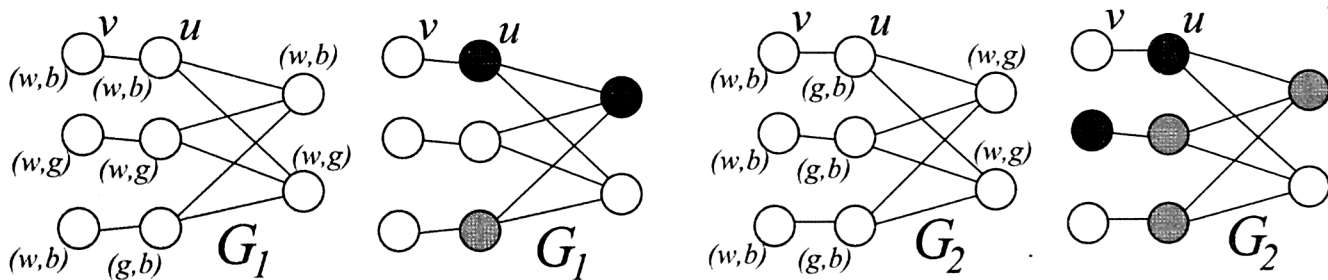
Figure 4.8: Two graphs that realize D_6 with example lists and list-distinguishing colorings using white, black, and grey.

Suppose lastly that G realizes D_{10} and that there is no vertex u in G such that $St(u) = \langle \sigma^j \rangle$ or $|O(u)| \geq 6$. If there exists a $u \in V(G)$ such that $St(u) = \langle \tau \sigma^i \rangle$, then $|O(u)| = 10$, therefore we may assume that every vertex u in G has $St(u) = \langle \sigma^j, \tau \sigma^i \rangle$. Given this stabilizer, it is not hard to show that for every vertex u in G, $|O(u)| = 1, 2$ or 5 with the Orbit/Stabilizer Theorem.

Furthermore, as every orbit of order 5 in $V(G)$ has the form $O(u) = \{u, u\sigma, \ldots, u\sigma^4\}$ and stabilizer $\langle \sigma^5, \tau \sigma^j \rangle$ for some j, it is not difficult to see

that the action of D_{10} on any such orbit of order five can be viewed precisely as the action of D_5 on the vertices of C_5. We observe that this implies that G is 3-list-distinguishable, as then we may color each orbit of order five such that three vertices receive distinct colors and also color the orbits of order two so that both vertices receive distinct colors. Keeping in mind the action of $Aut(G)$ on $O(u)$, we can see that this is a list-distinguishing coloring.

Again we can easily 2-list-distinguish G if there are no orbits of order 5, so we first assume there is exactly one orbit of order 5. Specifically, let u be a vertex such that $|O(u)| = 5$ and recall that necessarily $O(u) = \{u, u\sigma, \ldots, u\sigma^4\}$. Suppose as well for every $x \in O(u)$ there is some $\phi_x \in St(x)$ such that the following hold:

1. ϕ_x interchanges the pairs $(x\sigma, x\sigma^4)$ and $(x\sigma^2, x\sigma^3)$, and

2. ϕ_x fixes all of $V(G) - O(u)$.

As in the previous case, we claim that the existence of these ϕ_x implies that G is not 2-distinguishable. In any 2-coloring of $O(u)$, there is some vertex x such that the pairs $(x\sigma, x\sigma^4)$ and $(x\sigma^2, x\sigma^3)$ are monochromatic. However, then the automorphism ϕ_x described above is nontrivial and color-preserving, regardless of how the remainder of G is colored. It follows that G is not 2-list-distinguishable and, by our above observation, that G must therefore be 3-list-distinguishable.

Suppose therefore, without loss of generality, that there is no such $\phi_u \in Aut(G)$ and assign lists of length two to $V(G)$. We color each orbit of length two with distinct colors, and if possible assign distinct colors to three vertices

in $O(u)$. If it is not possible to color three vertices with distinct colors, then each vertex in $O(u)$ must have the same list, $\{c_1, c_2\}$. We will then color u with color c_1 and color $u\sigma$, $u\sigma^2$, $u\sigma^3$ and $u\sigma^4$ with color c_2. Given the action of D_{10} on $O(u)$, the only nontrivial automorphisms that would fix this coloring of $O(u)$ necessarily fix u and exchange the pairs $(u\sigma, u\sigma^4)$ and $(u\sigma^2, u\sigma^3)$. By the assumption that there is no ϕ_u satisfying conditions (1) and (2), such an automorphism must exchange the vertices in an orbit of order two and is therefore not color-preserving. It follows that G is 2-list-distinguishable.

Now assume there is more than one orbit of order 5 and assign lists of length 2 to the vertices of G. Choose vertices u and v in distinct orbits such that $|O(u)| = |O(v)| = 5$. If it is possible to color three vertices in $O(u)$ with distinct colors, then such a coloring can be extended to a 2-list-coloring of G (see G_1 in Figure 4.9). If it is not possible to color three vertices with distinct colors, then each vertex in $O(u)$ must have the same list, $\{c_1, c_2\}$. We then color two vertices, v and w, with c_1 and the rest with color c_2. If g, a nonidentity element of D_{10}, fixes this coloring of $O(u)$, then either $g \in St(v) \cap St(w)$ or g interchanges v and w. However, since $|O(v)| = 5$, g must exchange two elements in $O(v)$, call them v' and v''. Assigning v' and v'' distinct colors from their respective lists serves to 2-list-distinguish G (see G_2 in Figure 4.9). Therefore, $D(G) = D_\ell(G) = 2$. ∎

As a consequence of the proof of Theorem 4.7, we obtain the following, which tells us when the (traditional) distinguishing number is 2 and when it is 3 for the small cases. This information was vital for determining that $D(G) = D_\ell(G)$ for all graphs G that realize the dihedral group.
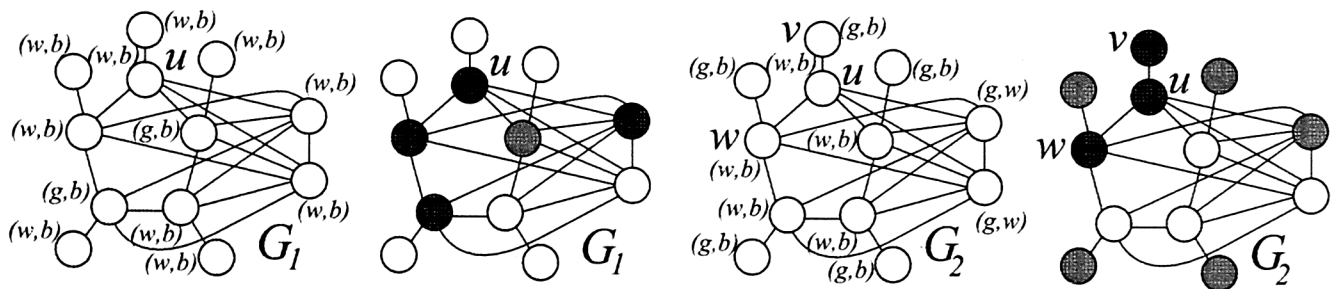
Figure 4.9: Two graphs that realize D_{10} with example lists and list-distinguishing colorings using white, black, and grey.

Theorem 4.13 *Let G be a graph realizing D_n such that $V(G)$ has no orbit of order greater than five, and also such that G does not satisfy the hypotheses of Lemma 4.8 or Lemma 4.10. Then $D(G) = 3$ if and only if G has exactly one nontrivial orbit O satisfying one of the following:*

1. *$n \in \{3, 4, 5\}$ and $|O| = n$, or*

2. *$n = 6$, $|O| = 3$, and for each $x \in O$ there is some $\phi_x \in St(x)$ such that the following hold:*

(a) *ϕ_x interchanges the two vertices in $O(u) - x$, and*

(b) *ϕ_x fixes all of $V(G) - O(u)$, or*

3. *$n = 10$, $|O| = 5$, and for each $x \in O$ there is some $\phi_x \in St(x)$ such that the following hold:*

(a) *ϕ_x interchanges the pairs $(x\sigma, x\sigma^4)$ and $(x\sigma^2, x\sigma^3)$, and*

(b) *ϕ_x fixes all of $V(G) - O(u)$.*

5. List-Distinguishing Cartesian Products

In this chapter we look at the list-distinguishing number of Cartesian product graphs. We first show that most Cartesian product graphs have list-distinguishing number 2. Then we show that $D_\ell(G) = D(G)$ for the Cartesian product of two cycles, and lastly we give the list-distinguishing number of the Cartesian product of two complete graphs.

5.1 Motion Lemma

A graph is called *prime* (with respect to the Cartesian product) if it cannot be represented as the Cartesian product of two nontrivial graphs. Two graphs G and H are *relatively prime* (with respect to the Cartesian product) if there is no nontrivial graph that is a factor of both G and H. For example if two graphs are not isomorphic and prime, then they are relatively prime. Finally, a *fiber* $G_i^{(g_1,\ldots,g_k)}$ of $G_1 \square \cdots \square G_k$ is the subgraph induced by the vertex set $\{(g_1, g_2, \ldots, g_{i-1}, x, g_{i+1}, \ldots, g_k) | x \in G_i\}$.

The automorphism group of Cartesian products is well understood. If $G = (G_1 \square G_2)$ and $\phi \in Aut(G_1)$, then $\phi' : V(G) \mapsto V(G)$ defined by $\phi' : (u_i, v_j) \mapsto (\phi u_i, v_j)$ is an automorphism of G. If $G_1 = G_2$, then $\alpha : (u_i, v_j) \mapsto (v_j, u_i)$ is also an automorphism of G. In fact, the automorphisms of a factor and permutations of isomorphic factors generate $Aut(G)$ [32].

The distinguishing number of Cartesian products has been extensively studied. In [6] Bogstad and Cowen determined that $D(Q_d) = 2$ if $d \geq 4$, where Q_d

denotes the d-dimensional hypercube. In 2005 Albertson generalized this result to connected prime graphs [1], using a probabilistic result from Russell and Sundaram [45].

Let $\phi \in Aut(G)$ and define $m(\phi) = |\{v \in G : \phi(v) \neq v\}|$. We refer to $m(\phi)$ as the motion of ϕ. The motion of a graph G, $m(G)$ is defined as

$$m(G) = \min_{\substack{\phi \in Aut(G) \\ \phi \neq id}} m(\phi).$$

If we decompose the automorphism ϕ into a product of disjoint cycles $\phi = (v_{11}v_{12} \ldots v_{1\ell_1})(v_{21}v_{22} \ldots v_{2\ell_2}) \ldots (v_{k1}v_{k2} \ldots v_{k\ell_k})$, then the *cycle norm* of ϕ is the quantity

$$c(\phi) = \sum_{i=1}^{k}(\ell_i - 1).$$

The *cycle norm* of a graph G is defined by

$$c(G) = \min_{\substack{\phi \in Aut(G) \\ \phi \neq id}} c(\phi).$$

The lemma by Russell and Sundaram, referred to as the *motion lemma*, showed if the motion of G is sufficiently large then the distinguishing number is small. We now show that the motion lemma works the same for list-distinguishing colorings. The proof of the proposition below follows closely with the proof in [45].

Proposition 5.1 *If $c(G) \ln d > \ln |Aut(G)|$ then G is d-list-distinguishable.*

Proof: Let G be a graph. For each $v \in V(G)$, we assign list L_v such that $|L_v| = d$. We select a color for each vertex v independently and uniformly from L_v and call this coloring χ. Let $\phi \in Aut(G)$ such that $\phi \neq e$. The probability

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that ϕ preserves χ is

$$Pr_{\chi}[\forall v, \chi(v) = \chi(\phi(v))] = (\frac{1}{d})^{c(\phi)} \leq (\frac{1}{d})^{c(G)}.$$

All together subadditivity gives that,

$$Pr_{\chi}[\exists \phi \neq e, \forall v, \chi(v) = \chi(\phi(v))] \leq |Aut(G)|(\frac{1}{d})^{c(G)}.$$

By the hypothesis, this quantity is less than 1, so there exists a list-coloring χ such that for all $\phi \neq e$, there exists a vertex, v, such that $\chi(v) \neq \chi(\phi(v))$. ■

Albertson found that $D(G^r) = 2$ if $r \geq 4$ and if $|V(G)| \geq 5$ then $D(G^r) = 2$ when $r \geq 3$ [1]. To proof this statement he observed that any automorphism of a graph G with n vertices is an automorphism of K_n and proved the following two theorems.

Theorem 5.2 *[1]* $|Aut(K_n^r)| = r!(n!)^r$

Theorem 5.3 *[1]* *If $n \geq 3$, then $m(K_n^r) = 2n^{e-1}$*

Albertson found that $r!(n!)^r < 2^{n^{r-1}}$ when $r \geq 4$ or when $n \geq 5$ and $r \geq 3$, and thus this is when $D(K_n^r) \leq 2$. In the same way, we can combine Proposition 5.1, Theorem 5.2, and Theorem 5.3 to arrive at the following corollary.

Corollary 5.4 *If G is a connected graph that is prime with respect to the Cartesian product, then $D_\ell(G^r) = 2$ if $r \geq 4$ and if $|V(G)| \geq 5$ then $D_\ell(G^r) = 2$ when $r \geq 3$.*

Albertson points out, using [28] and [34], that almost all graphs satisfy the hypotheses of Corollary 5.4.

5.2 Products of Cycles

We now limit our work to the Cartesian product of two graphs, since these are not covered under Corollary 5.4. In [37] it was found that $D(G^r) = 2, r \geq 3$ for any connected graph $G \neq K_2$ by Klavžar and Zhu. Imrich and Klavzar rounded out these investigations when they showed that $D(G^r) = 2$ when $r \geq 2$ and $G \neq K_2, K_3$ the same year [35].

We now turn our attention to the Cartesian product of two cycles, also known as the toroidal grid. We label a vertex of $C_n \square C_m$ as (u_i, v_j) if it is in fibers C_m^i and C_n^j, but for simplicity, we denote $L((u, v))$ as $L(u, v)$. We also define $S_c(G) = \{v \in V(G) \mid c \in L(v)\}$ and will write $S_c(G) = S_c$ if the context is clear. If $n \neq m$, the automorphism group of $C_n \square C_m$ is generated by the Cartesian product of the generators of C_n and C_m. This leads us to the following elementary lemma about the action of an automorphism of a cycle factor C_n, which will be used in subsequent proofs.

Lemma 5.5 *Let $G = (C_n \square C_m)$ such that $n \neq m$. Let $\phi \in Aut(C_n)$, $\psi \in Aut(C_m)$ and $\phi' \in Aut(C_n \square C_m)$ such that $\phi' : (u_i, v_j) \mapsto (\phi u_i, \psi v_j)$. If $\phi' : (u_i, v_j) \mapsto (u_k, \psi v_j)$, then $\phi' : (u_i, v_p) \mapsto (u_k, \psi v_p)$ for all $p \in \{1, \ldots, m\}$.*

Proof: Let $G = (C_n \square C_m)$, $n \neq m$, and $(u_i, v_j) \in V(G)$. Let $\phi \in Aut(C_n)$, $\psi \in Aut(C_m)$ and $\phi' \in Aut(C_n \square C_m)$ such that $\phi' : (u_i, v_j) \mapsto (\phi u_i, \psi v_j)$. Let $\phi' : (u_i, v_j) \mapsto (u_k, \psi v_j)$. This implies that $\phi(u_i) = (u_k)$. Since $Aut(C_n) = D_n = < \sigma_n, \tau_n | \sigma_n^{2n} = \tau_n^2 = 1, \sigma_n \tau_n = \tau_n \sigma_n^{-1} >$, we can write $\phi' = (\sigma_n^a \tau_n^b, \psi)$ where $a \in \{0, \ldots, 2n - 1\}$ and $b \in \{0, 1\}$. Let $(u_i, v_p) \in V(G)$. Now $\phi'(u_i, v_p) = (\sigma_n^a \tau_n^b, \psi)(u_i, v_p) = (\sigma_n^a \tau_n^b(u_i), \psi v_p)$. Since $\phi u_i \mapsto u_k$, $(\sigma_n^a \tau_n^b(u_i), \psi v_p) = (u_k, \psi v_p)$. Since p was arbitrary, this is true for all $p \in \{1, \ldots, m\}$. ∎
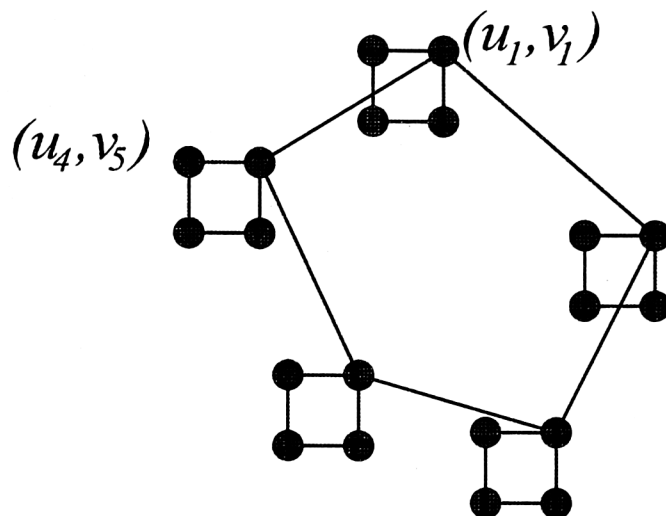
Figure 5.1: Let $\phi' : (u_i, v_j) \mapsto (\phi u_i, \sigma_5^4 v_j) \in Aut(C_4 \square C_5)$ to illustrate Lemma 5.5. If $(u_1, v_1) \mapsto (u_4, v_5)$ in ϕ', then $(u_1, v_2) \mapsto (u_4, v_1)$, $(u_1, v_3) \mapsto (u_4, v_2)$, $(u_1, v_4) \mapsto (u_4, v_3)$, and $(u_1, v_5) \mapsto (u_4, v_4)$.

To prove that $D(C_n \square C_m) = D_\ell(C_n \square C_m)$ for all n and m, we use the following lemma, which says that if we can 2-list-distinguish any fiber of $C_n \square C_m$ in a specific way then the entire graph can be 2-list-distinguished.

Lemma 5.6 *Let $G = C_n \square C_m$, and assign a list $L(u_i, v_j)$ of size 2 to each $(u_i, v_j) \in V(C_n \square C_m)$. If there exists a fiber that can be L-list-distinguished with a vertex that has a unique color in that fiber, then G can be L-list-distinguished.*

Proof: Let $G = C_n \square C_m$, and assign a list $L(u_i, v_j)$ such that $|L(u_i, v_j)| = 2$ to each $(u_i, v_j) \in V(C_n \square C_m)$. Assume we can color C_n^1 using a unique color on the vertex (u_1, v_1).

Case 1: Suppose $V(C_n^2) \subseteq S_{c_1}$.

Color each vertex $(u_i, v_2) \in C_n^2$ with c_1. Assign list $L'(u_x, v_y) = L(u_x, v_y) - c_1$ to all uncolored vertices, and L'-list-color these vertices arbitrarily (see Case 1 in Figure 5.2).

Case 2: There exist vertices $(u_i, v_2), i \neq 1$ and $(u_1, v_j), j \neq 1$ such that $(u_i, v_2), (u_1, v_j) \notin S_{c_1}$.

For all uncolored vertices, assign list $L'(u_x, v_y) = L(u_x, v_y) - c_1$. Color (u_i, v_m) with some $c_2 \in L'(u_i, v_m)$, color (u_i, v_2) with some $c_3 \in L'(u_i, v_2)$ such that $c_3 \neq c_2$, and L'-list-color the rest of the vertices. Since they are the only two fibers with a vertex colored c_1, C_n^1 must map to C_m^1 in any non-trivial color-preserving automorphism; furthermore, C_n^1 has been list-distinguished, so there is at most one automorphism, α, that could map C_n^1 to C_m^1. Assume that $\alpha(u_i, v_1) \mapsto (u_1, v_j)$, and let the vertex (u_i, v_1) be colored c_k. The vertex $(u_1, v_j) \notin S_{c_1}$, so recolor (u_1, v_j) with $c_a \in L'(u_1, v_j)$ such that $c_a \neq c_k$ (see Case 2 in Figure 5.2).

Case 3: There exists a vertex (u_i, v_2), $i \neq 1$, such that $(u_i, v_2) \notin S_{c_1}$, and $(u_1, v_j) \in S_{c_1}$ for all $j \neq 1$.

Color (u_1, v_2) with c_1, and assign list $L'(u_x, v_y) = L(u_x, v_y) - c_1$ to all uncolored vertices. Let vertex (u_i, v_1) be colored c_p, and color (u_i, v_2) with $c_b \in L'(u_i, v_2)$ such that $c_b \neq c_p$. Now L'-list-color the rest of the vertices (see Case 3 in Figure 5.2).

We claim that in each of the three cases above the coloring is list-distinguishing. First consider an automorphism of the form $(\sigma_n^a \tau_n^b, e)$, $\sigma_n^a \tau_n^b \neq e$, which maps (u_t, v_1) to (u_r, v_1) for some $t \neq r$ by Lemma 5.5. The graph induced by the vertices of C_n^1 has been assigned a distinguishing coloring. Thus an automorphism of this form does not preserve colors.

Also by Lemma 5.5, the automorphism $(e, \sigma_m^c \tau_m^d)$, $\sigma_m^c \tau_m^d \neq e$, maps either (u_i, v_2) to $(u_i, v_j), 2 \neq j$ for all $i \in \{1, \ldots, n\}$, or (u_1, v_1) to $(u_1, v_j), 1 \neq j$.

However in case 1, $n - 1$ vertices of C_n^2 are colored c_1, and this is the only fiber with such a coloring. The vertex (u_1, v_1) is colored c_1 while the rest of the fiber C_n^1 is not; again this is the only fiber with such a coloring. In Case 2, (u_1, v_1) is the only vertex colored c_1, and we know that (u_i, v_2) must map to (u_i, v_m), which are colored differently. In Case 3, (u_1, v_1) and (u_1, v_2) are the only two vertices colored c_1, so any non-trivial automorphism must interchange (u_1, v_1) and (u_1, v_2), which will also interchange (u_i, v_1) and (u_i, v_2), which have different colors. Therefore in each of these cases, automorphisms of this type are not color-preserving.

The automorphism $(\sigma_n^a \tau_n^b, \sigma_m^c \tau_m^d)$ is equal to $(e, \sigma_m^c \tau_m^d) \circ (\sigma_n^a \tau_n^b, e)$. In Case 1, the automorphism $(\sigma_n^a \tau_n^b, e)$ will map (u_1, v_1) to (u_j, v_1) for some $j \in \{1 \ldots n\}$ and (u_i, v_2) to (u_k, v_2) for all $i \in \{1, \ldots, n\}$. The $n - 1$ vertices of C_n^2 are colored with c_1 and the vertex (u_1, v_1) is in fiber C_n^1, so as in the previous paragraph, $(\sigma_n^a \tau_n^b, e)$ is not color-preserving. Similarly, $(\sigma_n^a \tau_n^b, e)$ is not color-preserving in Cases 2 and 3. Therefore in each of these cases, automorphisms of this type are not color-preserving.

If $n = m$ consider the automorphism α, such that $\alpha(u_i, v_1) \mapsto (u_1, v_j)$ for some i and j, which would map C_m^1 to C_n^1. In Case 1, C_n^2 has $n - 1$ vertices colored c_1 and there is no C_m fiber with this number of vertices colored c_1. In Case 2, α is not color-preserving by coloring (u_i, v_1) and (u_1, v_j) differently. In Case 3, C_m^1 has two vertices colored c_1 and no C_n fiber has more than one vertex colored c_1. Therefore in each of these cases, α automorphisms are not color-preserving.

In every case, colors can be chosen from the list of size 2 to list-distinguish $C_n \square C_m$. If there exists a fiber that can be L-list-distinguished, $|L(v)| = 2$ for all $v \in V(G)$, with a vertex that has a unique color for that fiber, then G can be L-list-distinguished. ∎

In Figure 5.2, $\sim c$ refers to any color that that is not c and vertices colored black represent the vertices colored c_1 in the proof of Lemma 5.6. We now use Lemma 5.6 to 2-list-distinguish all graphs that are the Cartesian product of two cycles except $C_3 \square C_3$.
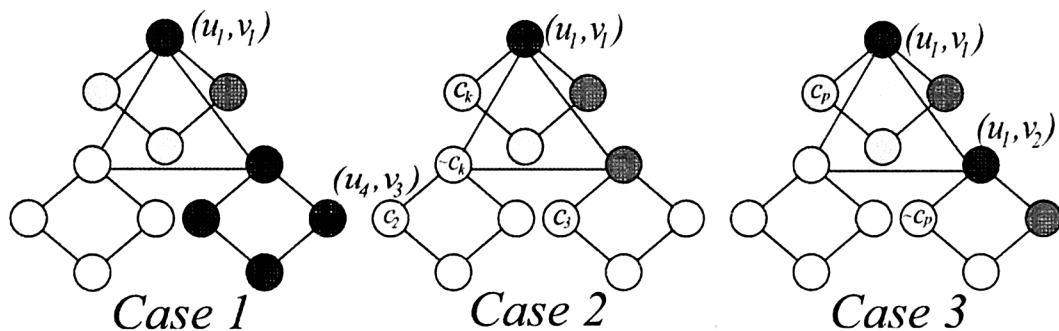
Figure 5.2: Examples of colorings described in Lemma 5.6. Vertices represented by open circles cannot be colored black.

Lemma 5.7 *Let $n \geq 3$ and $m \geq 3$ such that at most one of n and m is 3. Then*
$$D_\ell(C_n \square C_m) = 2.$$

Proof: Assign a list $L(u_i, v_j)$ such that $|L(u_i, v_j)| = 2$ to each $(u_i, v_j) \in V(C_n \square C_m)$. If $|\bigcup L(u_i, v_j)| = 2$, then color the vertices as in [6] if $n = m = 4$ or as in [35] otherwise. If not, then there must be two vertices whose lists are different. Since $C_n \square C_m$ is connected, there exist two adjacent vertices with different lists, and these two vertices are either on the same C_n or C_m fiber. Without loss of generality, assume they are the vertices (u_1, v_1) and (u_2, v_1) on C_n^1. Color the vertex (u_n, v_1) with $c_3 \in L(u_n, v_1)$; color vertices (u_1, v_1) and

(u_2, v_1) from their lists such that the colors are different and both not c_3, say c_1 and c_2 respectively, which is possible since their lists are not the same. Color the rest of the vertices of C_n^1 anything from their list other than c_1.

Since (u_1, v_1) is the only vertex with color c_1, any color-preserving automorphism on the fiber C_n^1 must map (u_1, v_1) to itself. Therefore, either $(u_n, v_1) \mapsto (u_2, v_1)$ or $(u_n, v_1) \mapsto (u_n, v_1)$. In the former case, the automorphism does not preserve the colors. The latter case is the trivial automorphism, so the coloring is an L-list-distinguishing coloring on C_n^1 that uses a unique color c_1. By Lemma 5.6, we know that we can extend this coloring to a L-list-distinguishing coloring of $C_n \square C_m$. Therefore, $D_\ell(C_n \square C_m) = 2$ if at most one of n and m is 3.

∎

Lemma 5.8 $D_\ell(C_3 \square C_3) = D(C_3 \square C_3) = 3$.

Proof: Assign list $L(u_i, v_j)$ such that $|L(u_i, v_j)| = 3$ to each $u_i, v_j \in V(C_3 \square C_3)$. Color the vertices (u_i, v_1), $i \in \{1, 2, 3\}$ from their list such that each vertex received a unique color. Let vertex (u_i, v_1) for $i \in \{1, 2, 3\}$ be colored c_i, and we will not use the color c_1 on any other vertex. If one of the vertices (u_1, v_2) and (u_1, v_3) has a color that is not c_1, c_2, or c_3 in its list, say c_4, then use that color. Then color the other anything from its list that is not c_1 nor c_4 (see G in Figure 5.3). If both of their lists are $\{c_1, c_2, c_3\}$, then color them both c_2 (see H in Figure 5.3). Lastly, color (u_3, v_2) and (u_3, v_3) from their list such that the colors are different and both not c_1, and then color the remaining vertices anything from their list that is not c_1.

By Lemma 5.5 and since C_3^1 has a list-distinguishing coloring, there is no color-preserving automorphism of the form $(\sigma_n^a \tau_n^b, e)$. Since (u_1, v_1) is the only vertex with color c_1, the only non-trivial automorphism of the form $(e, \sigma_m^c \tau_m^d)$ or $(\sigma_n^a \tau_n^b, \sigma_m^c \tau_m^d)$ would map (u_3, v_2) to (u_3, v_3). However, these vertices are colored differently. Lastly, the automorphism $\alpha : (u_i, v_j) \mapsto (v_j, u_i)$ must map (u_1, v_1) to itself. However, (u_2, v_1) and (u_3, v_1) are colored differently than (u_1, v_2) and (u_1, v_3). Thus the only color preserving automorphism is the identity, and $D_\ell(C_3 \square C_3) = D(C_3 \square C_3) = 3$.
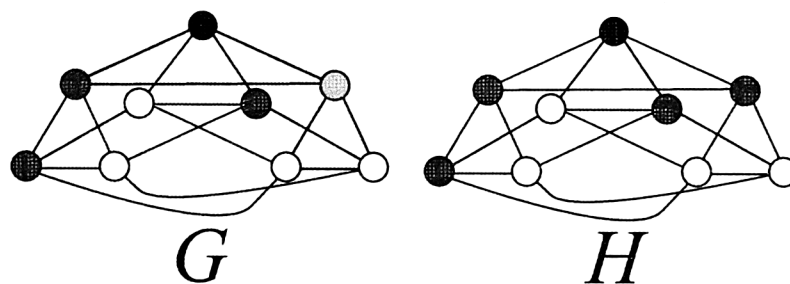
Figure 5.3: Examples of colorings described in Lemma 5.8.

The proof of the next theorem follows immediately from Lemma 5.7 and Lemma 5.8.

Theorem 5.9 $D_\ell(C_n \square C_m) = D(C_n \square C_m)$ *for all* $n, m \geq 3$.

5.3 Products of Complete Graphs

We now turn our attention to the Cartesian product of complete graphs. Imrich, Jerebic, and Klavžar [35] and Fischer and Isaak [22] independently determined the distinguishing number of Cartesian products of complete graphs.

Theorem 5.10 ([35]) *Let k, n, d be integers so that $d \geq 2$ and $(d-1)^k < n \leq d^k$. Then*

$$D(K_k \square K_n) = \begin{cases} d & \text{if } n \leq d^k - \lceil log_d k \rceil - 1; \\ \\ d+1 & \text{if } n \geq d^k - \lceil log_d k \rceil + 1. \end{cases}$$

If $n = d^k - \lceil log_d k \rceil$ then $D(K_k \square K_n)$ is either d or $d+1$ and can be computed recursively in $O(log^*(n))$ time.

Again, list-distinguishing this class of graphs has proved more difficult. Fisher and Isaak investigated this topic by coloring the cells of an $n \times m$ matrix such that the only row and column permutations that preserve colors is the identity. We present a proof that $D_\ell(K_3 \square K_4) = 2$; however, our method does not seem to generalize to $K_n \square K_m$. In the proof, we use the term *multiset* to mean a set where each member can have more than one membership (ie $\{3, 3, 4, 5\}$ is a multiset).

c_1	c_1	c_1	$\sim c_1$
c_2	c_2	$\sim c_2$	$\sim c_2$
c_3	$\sim c_3$	$\sim c_3$	$\sim c_3$

Figure 5.4: A list-distinguishing coloring of the matrix that represents $K_3 \square K_4$; c_1, c_2, and c_3 need not be distinct.

Proposition 5.11 $D_\ell(K_3 \square K_4) = 2$.

Proof: We consider the automorphism group of $K_3 \square K_4$ as the group generated by arbitrary cell permutations of a 3×4 matrix. Hence we assign lists of size two to the cells of a 3×4 matrix, color from those lists, and show that

the only composition of column and row permutations that is color-preserving is the identity.

Consider the coloring in Figure 5.4, and notice that after any permutation the four cells of row 1 will still comprise a row. Cell $(1,4)$ is the only cell in row 1 that is not colored c_1, thus column 4 must remain the fourth column after any color-preserving permutation. Similarly, cell $(2,3)$ is the only cell in the first three columns that is not colored c_2 in row 2. Thus any color-preserving permutation must fix column 3 as well. In the same way we can fix column 2, and now all of the columns must be fixed in any color-preserving permutation. Since the columns must be fixed, it is easy to see that no row permutations will preserve the colors. If we can color the matrix as in Figure 5.4, then we do so. If not then we consider each of the following cases.

Case 1: There does not exist a color that appears in three lists in any one row.

If there does not exist a row that contains two lists with a common color, then each row has eight distinct colors in the union of the lists. Color row 1 and 2 such that each cell in those two rows gets a unique color and color row 3 such that is not identical to row 1 or 2 (see matrix A in Figure 5.5). Since each row and column has a distinct multiset of colors, no row and column permutations will preserve colors. Thus this is a list-distinguishing coloring.

Without loss of generality, assume now that row 1 has the color c_1 in columns 1 and 2; color both of these cells with c_1. Now color the other two cells of row 1 with c_a and c_b such that $c_a \neq c_b$, which are both not equal to c_1. If no other row contains two lists with a common element, color each cell in the remaining

two rows uniquely, which we can do because the size of the union of the lists is at least 8 (see matrix B in Figure 5.5). In this case, each row and column has a different multiset of colors, thus this is a list-distinguishing coloring.

Now assume that either row 2 or 3 has two lists with a common color, c_2. Color both those cells with c_2 and consider the other two cells of that row. If both of these cells have lists $\{c_a, c_b\}$ then color them both with c_a (see matrix D in Figure 5.5); if not then color them differently such that at least one of them is not equal to c_a nor c_b (see matrix C in Figure 5.5). Now color the last row such that each cell contains a unique color in that row. We know we can do this because there is no row with a color that appears in three lists. In either coloring each row has a distinct multiset of colors, so no row permutations are color-preserving. Thus the columns are also fixed, so this is a list-distinguishing coloring.

c_1	c_2	c_3	c_4
c_5	c_6	c_7	c_8
c_1	c_6	c_7	c_4

A

c_1	c_1	c_a	c_b
c_1	c_2	c_3	c_4
c_5	c_6	c_7	c_8

B

c_1	c_1	c_a	c_b
c_a	c_2	c_2	c_d
c_1	c_2	c_3	c_4

C

c_1	c_1	c_a	c_b
c_a	c_2	c_2	c_a
c_1	c_2	c_3	c_4

D

c_1	c_1	c_1	$\sim c_1$
c_1	c_2	c_3	$\sim c_1$
c_4	c_5	c_6	$\sim c_1$

E

Figure 5.5: Examples of list-distinguishing colorings of the matrix that represents $K_3 \square K_4$ from the proof of Theorem 5.11.

Case 2: There exists a color c_1 that appears in three lists of a row but there is not a color that appears twice in the subset of the three columns in another row.

Assume that c_1 appears in the first three cells of row 1, color each of these cells with c_1, and color the fourth cell with a color from its list that is not c_1. Since no color appears twice in the first three cells of row 2 or 3, each row has six

distinct colors appearing in the lists of the first three cells. Thus we can color each of these six cells uniquely from their lists. Now color the remaining two cells anything from their list that is not c_1 (see matrix E in Figure 5.5). Again, the set of colors for each row and column is distinct, so this is a list-distinguishing coloring.

In every case we are able to list-distinguish the graph, thus $D_\ell(K_3 \square K_4) = 2$.

■

6. Concluding Remarks and Future Directions

In this final chapter, we discuss area for future study based on the work in this thesis. Again, we examine the two main areas of this thesis separately.

6.1 Interval Representations

In chapter 2 we give two characterizations for k-trees that have interval p-representations, one being a forbidden subgraph characterization for 2-tree interval p-graphs. In [11] Brown gives a characterization of interval p-graphs based on an ordering of complete r-partite subgraphs (see Theorem 1.2) and a forbidden subgraph characterization of cycle-free interval p-graphs (see Theorem 1.1). However, a complete forbidden subgraph characterization of interval p-graphs appears to be quite difficult, even when restricting p. Thus, we must narrow our focus.

One way to do narrow our focus is to restrict p and to examine a well-understood class of graphs. For example, we could narrow our focus to co-comparability graphs that have an interval 3-representation. A *cocomparability graph G* is one in which the edges of of the complement of G, denoted G^c, can be directed so that the resulting directed graph is transitive.

Another way to narrow our search for a forbidden subgraph characterization would be to examine ATE-free graphs. Asteroidal triples of edges were enough to characterize the k-trees that have interval p-representations; however, we know that ATEs do not completely characterize interval p-graphs. It would be

interesting to determine what subset of the ATE-free graphs have an interval p-representation.

In chapter 3 we give a characterization of probe interval 2-trees. The only forbidden subgraph characterization is of cycle-free probe interval graphs (see Theorem 1.3). There is still hope for the a forbidden subgraph characterization of probe interval 2-trees. However, which such a large obstruction set the proof may be a long case analysis. Moreover, the inconsistencies with the idea of minimal may need to be fixed.

We use the sparse spiny interior 2-lobster to characterize the probe interval 2-trees, and we believe this structure could be used to characterize k-trees with $k \leq 3$, with some tweaking of the definition. Thus we conjecture the following.

Conjecture 6.1 *Let G be a k-tree. G is a probe interval graph if and only if it is a spare spiny interior k-lobster.*

6.2 Symmetries

In chapter 4 we found that the list-distinguishing number is equal to the distinguishing number for graphs that realize the dihedral group. These findings bring us back to our original question.

Question 6.2 *Does there exist a graph G such that $D(G) < D_\ell(G)$?*

In the process of trying to answer Question 6.2, we may narrow our focus to finding the list-distinguishing number of classes of graphs for which the distinguishing number is known. In chapter 5 we investigated the list-distinguishing number of the Cartesian product of complete graphs. We were only able to prove that $D_\ell(K_3 \square K_4) = 2$. As this proof does not see to generalize, this is a ripe

area for investigation. Other classes of graphs for which one could determine the list-distinguishing number include trees and forests.

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